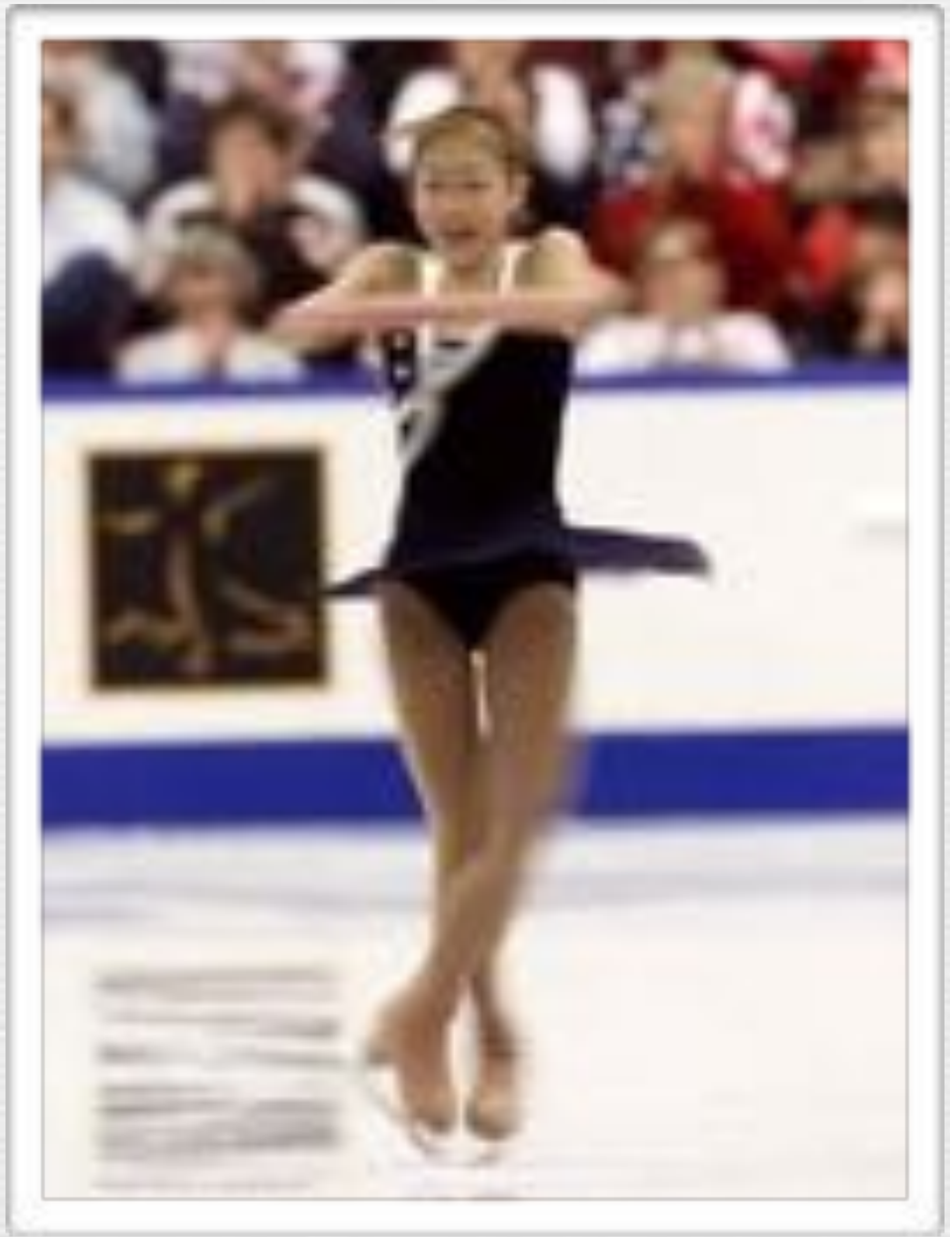
Rotational Mechanics

" We dance round in a ring and suppose, but the secret sits in the middle and knows." — Robert Frost

Why do you hold your arms out when trying to hold your balance (on a balance beam, tightrope, slack line, curb etc.)?

Why do figure skaters pull their arms and legs in when performing quick spins?

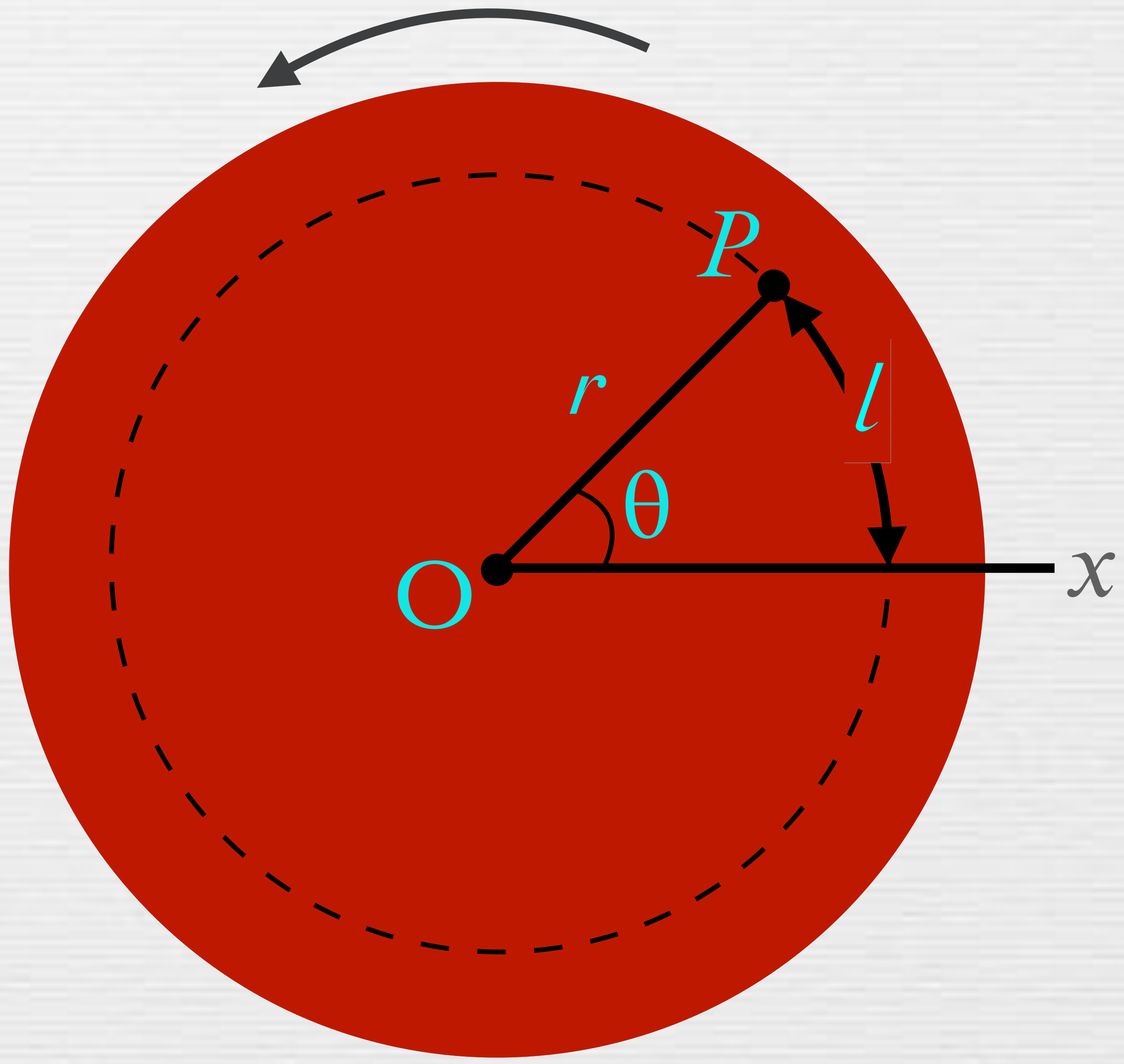


Rotational Motion

- Motion of a rigid body is usually broken up into *translational motion* of its center of mass and *rotational motion* about an axis of rotation
  - A rigid body is one which has definite and unchanging shape

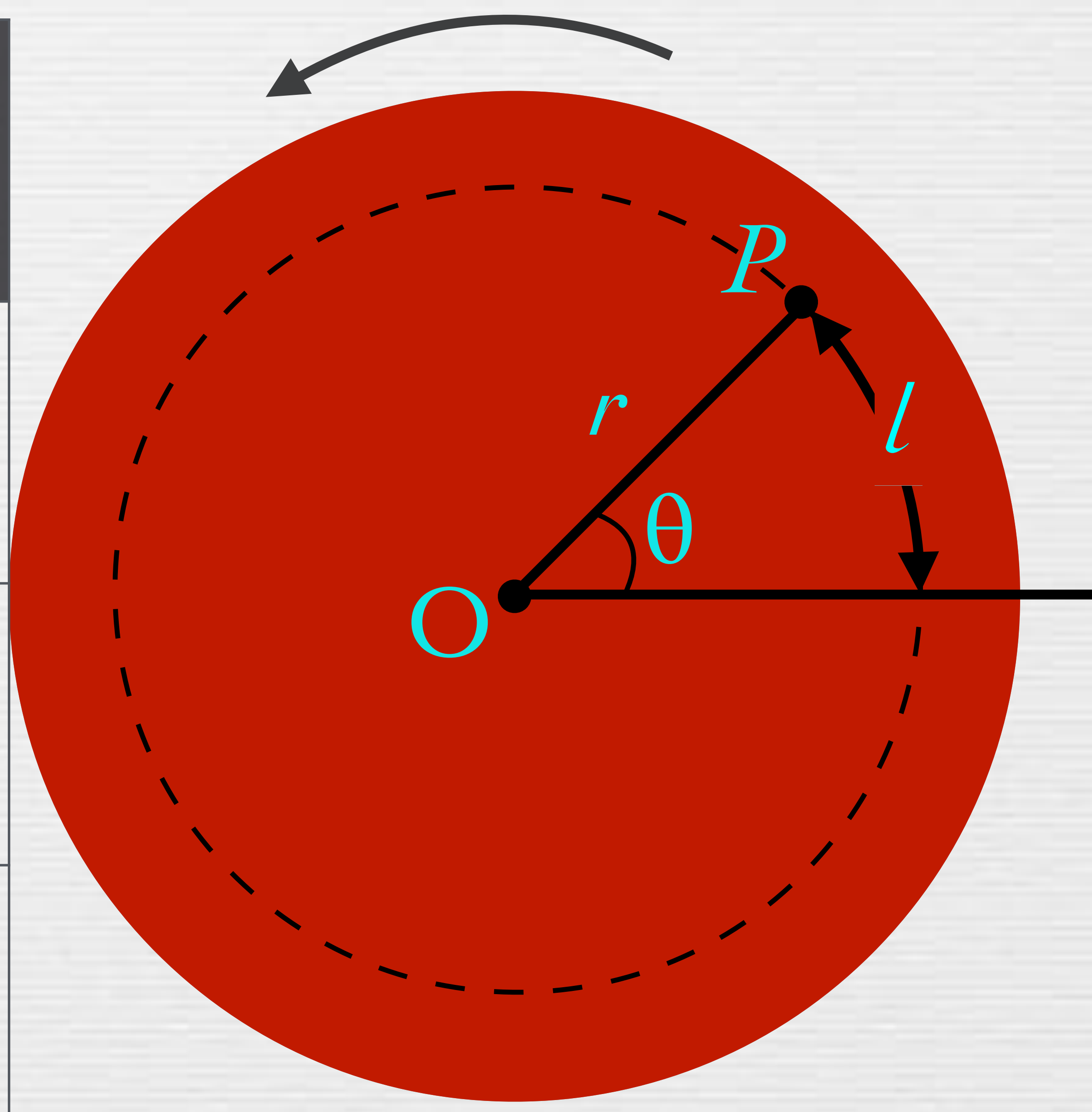
Angular Quantities

- Angular quantities are analogous to corresponding quantities in linear motion
  - Instead of asking how far and how fast an object travels, we can ask how much and how quickly it rotates



Angular Quantities

Quantity	Linear	Angular	Relationship
position	l in meters	θ in radians	$\theta = l/r$
velocity	v in m/s	ω in rad/s	$\omega = v/r$ $= \Delta\theta/\Delta t$
acceleration	a in m/s^2	α in rad/s^2	$\alpha = a/r$ $= \Delta\omega/\Delta t$



Note: 2π rad = 360°

Example 1

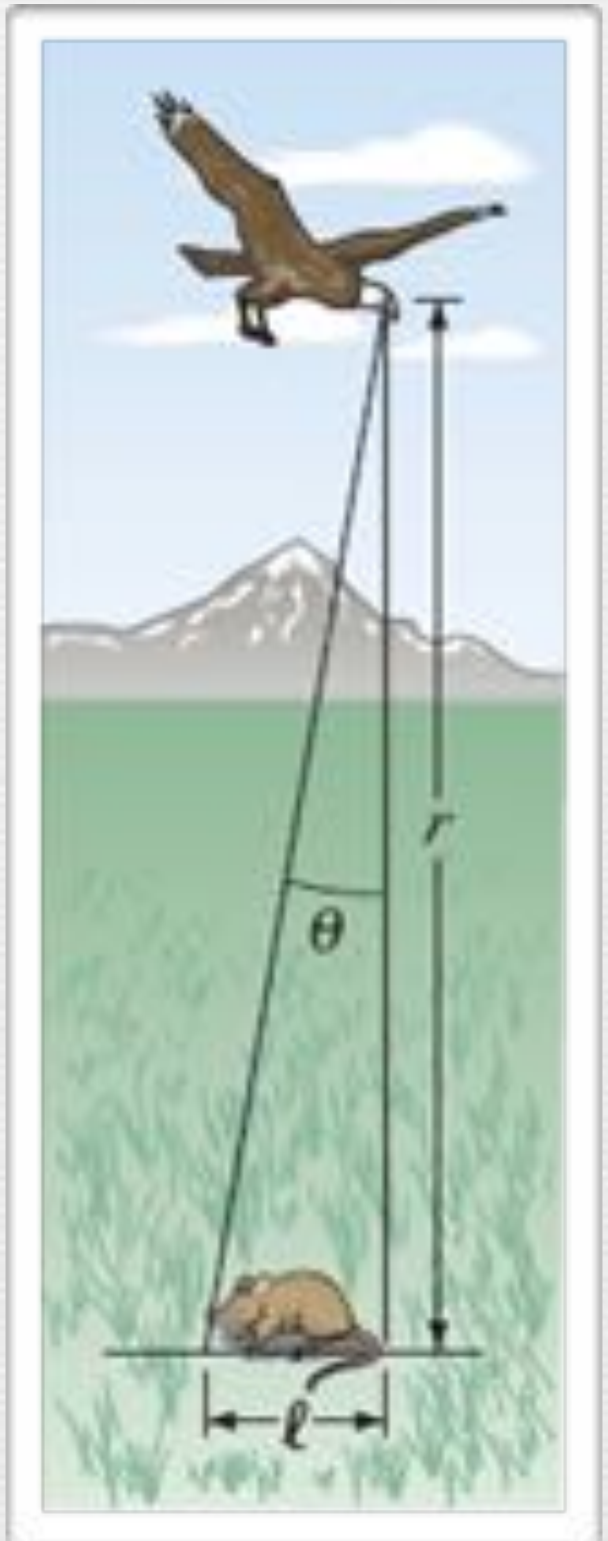
- A particular bird's eye can just distinguish objects that subtend an angle no smaller than about 3×10

  a) How many degrees is this?

  b) How small an object can the bird just distinguish when flying at a height of 100 m?

- *Ans. a)* $\theta = 0.017^{o}$

- *b)* $l = 3$ *cm*



Sanity Check

- **A rotating carousel has one child sitting on a horse near the outer edge and another child on a lion halfway out from the center.**

  a) **Which child has greater linear speed?**

  b) **Which child has greater angular speed?**

- ***Ans. a) the child on the horse***
- ***b) both are the same***

Circular Motion & Angular Quantities

- Centripetal acceleration in terms of angular velocity
  - $a_c = \omega^2 r$
- Frequency in terms of angular velocity
  - $\omega = 2\pi f$

Example 2

a) What is the linear speed of a child seated 1.2 m from the center of a steadily rotating merry-go-round that makes one complete revolution in 4.0 s?

b) What is her acceleration (tangential and centripetal)?

- *Ans. a)* $v = 1.9\ m/s$
- *b)* $a_t = 0;\ a_c = 3.0\ m/s^2$

Example 3

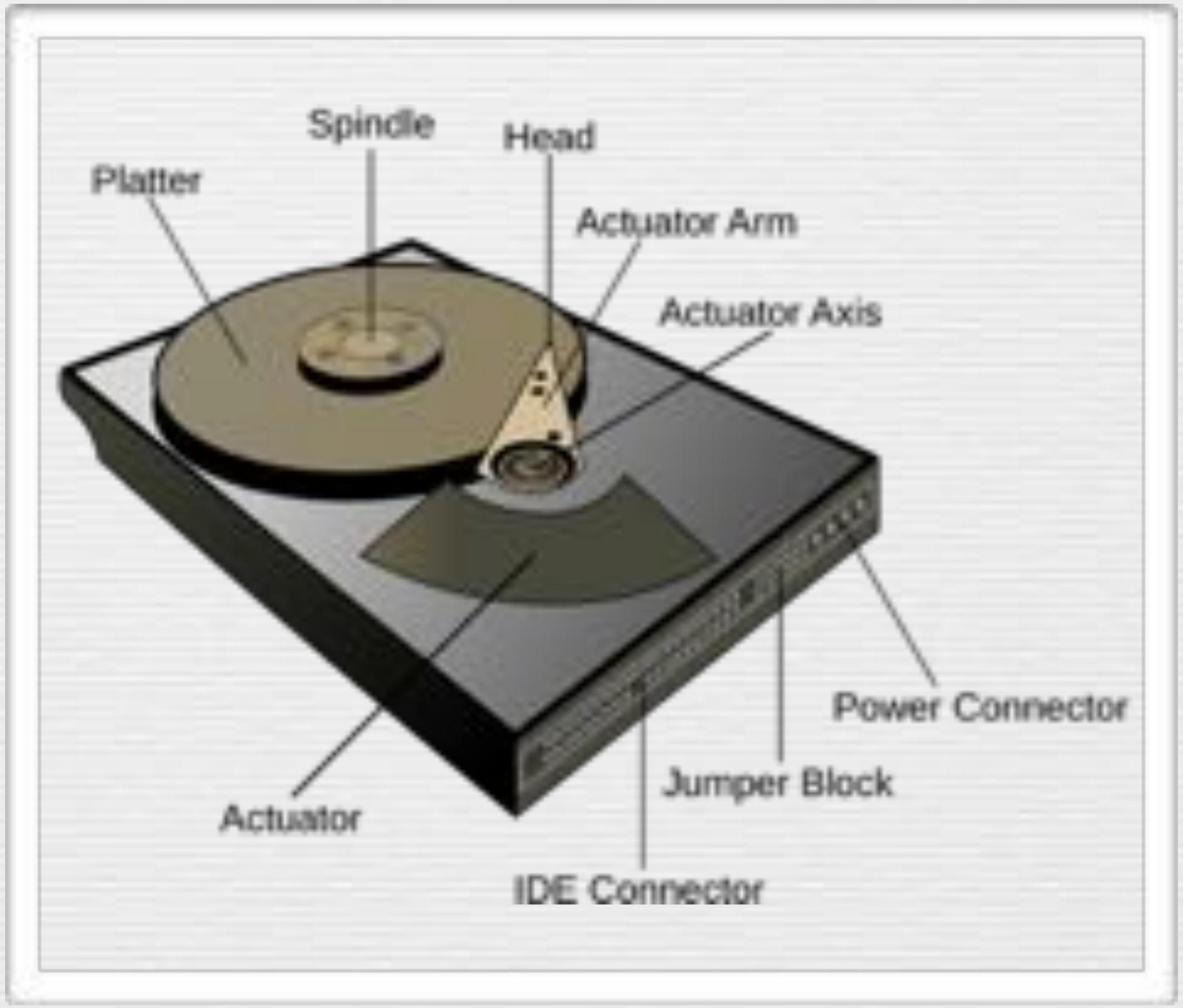
- The platter of the hard disk of a computer rotates at 5400 rpm.

a) What is the angular velocity of the disk?

b) If the reading head is of the drive is located 3.0 cm from the axis of rotation, what is the speed of the of the disk below it?

c) What is the linear acceleration of this point?

d) If a single bit requires 5 μm of length along the motion direction, how many bits per second can the writing head write when it is 3.0 cm from the axis?



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- *Ans. a)* $\omega = 570$ *rad/s*
- *b)* $v = 17$ *m/s*
- *c)* $a_c = 9700$ *m/s*^{2}
- *d)* 3.4×10^6 *bits per second = 425 Kbps*

Kinematic Equations

Angular	Linear
$\omega_f = \omega_i + \alpha\Delta t$	$v_f = v_i + a\Delta t$
$\Delta\theta = \omega_i\Delta t + \frac{1}{2}\alpha\Delta t^2$	$\Delta x = v_i\Delta t + \frac{1}{2}a\Delta t^2$
$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$	$v_f^2 = v_i^2 + 2a\Delta x$

Note: remember! the kinematic equations only work for *constant* acceleration, whether angular or linear

Example 4

- A centrifuge rotor is accelerated from rest to 20,000 rpm in 5.0 min.
- How many revolutions has it turned through in this time?
  - *Ans.* $\alpha = 7.0\ rad/s^2$
  - $\Delta\theta = 3.15\times10^5\ rad = 5.0\times10^4\ rev.$

Rolling Motion

- A bicycle slows down uniformly from
  a distance of 115 m. Each wheel and tire has an overall diameter of 68.0 cm. Determine

a) the angular velocity of the wheels at the initial instant

b) the total number of revolutions each wheel rotates in coming to rest

c) the angular acceleration of the wheel

d) the time it took to come to a stop

Rolling Motion

- *A bicycle slows down uniformly from v_0 = 8.40 m/s to rest over a distance of 115 m. Each wheel and tire has an overall diameter of 68.0 cm. Determine*

a) the angular velocity of the wheels at the initial instant

- $\omega_0 = v_0/r$
- ω_0 = (8.40 m/s)/(0.340 m)
- ω_0 = 24.7 rad/s

Rolling Motion

- *A bicycle slows down uniformly from v_0 = 8.40 m/s to rest over a distance of 115 m. Each wheel and tire has an overall diameter of 68.0 cm. Determine*

b) the total number of revolutions each wheel rotates in coming to rest

- Revs = d/C
- Revs = $d/(2\pi r)$
- Revs = (115 m)/(2π•0.340 m)
- Revs = 53.8 rev

Rolling Motion

- *A bicycle slows down uniformly from v_0 = 8.40 m/s to rest over a distance of 115 m. Each wheel and tire has an overall diameter of 68.0 cm. Determine*

c) the angular acceleration of the wheel

- $\alpha = (\omega_1^2 - \omega_0^2)/(2\Delta\theta)$
- $\alpha = (0 - (24.7 \text{ rad/s})^2)/(2 \cdot 2\pi \cdot 53.8 \text{ rev})$
- $\alpha = -0.902 \text{ rad/s}^2$

Rolling Motion

- *A bicycle slows down uniformly from $v_0 = 8.40$ m/s to rest over a distance of 115 m. Each wheel and tire has an overall diameter of 68.0 cm. Determine*

c) the time it took to come to a stop

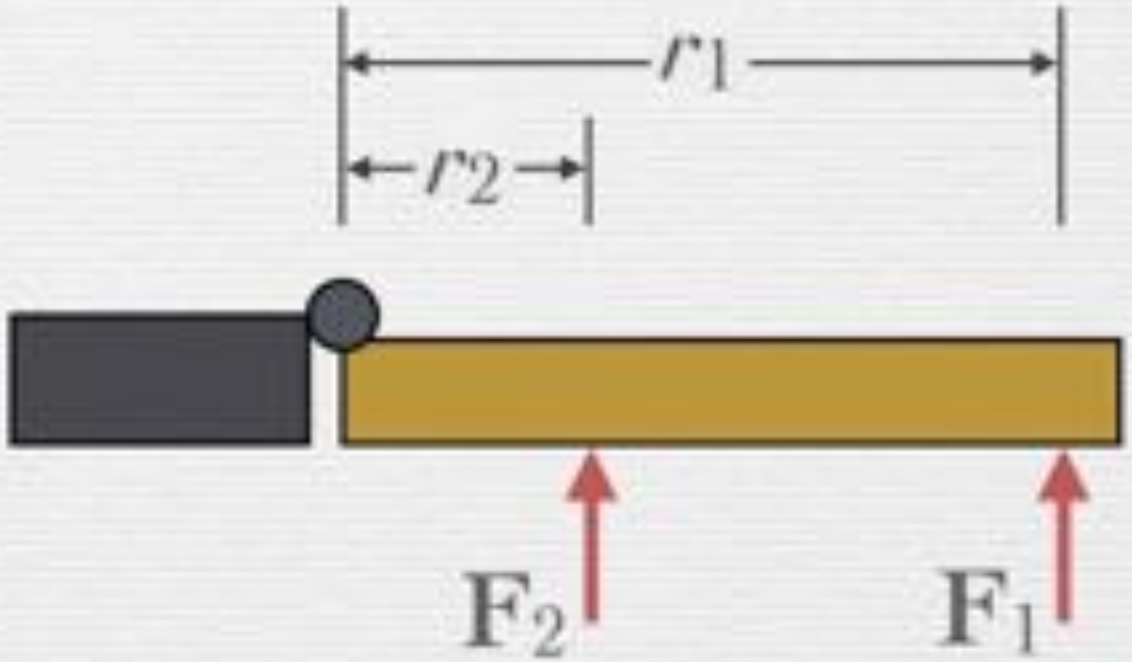
- $t = (\omega_1 - \omega_0)/\alpha$
- $t = (0 - 24.7 \text{ rad/s})/(-0.902 \text{ rad/s}^2)$
- $t = 27.4$ s

Torque

- Rotational kinematics — *how* things rotate
- Rotational dynamics — *why* things rotate
  - To make an object rotate, we need a force
  - But the direction of the force, and where we apply it, matters

Torque

- Apply force $\mathbf{F}_1$
  - The bigger the force, the more quickly the door opens
- Apply the same force closer to the hinge, at $\mathbf{F}_2$
  - Door will not open as quickly
- The angular acceleration of the door is proportional to the magnitude of the force applied *and* the distance that force is from the axis of rotation
  - That distance is called the lever arm

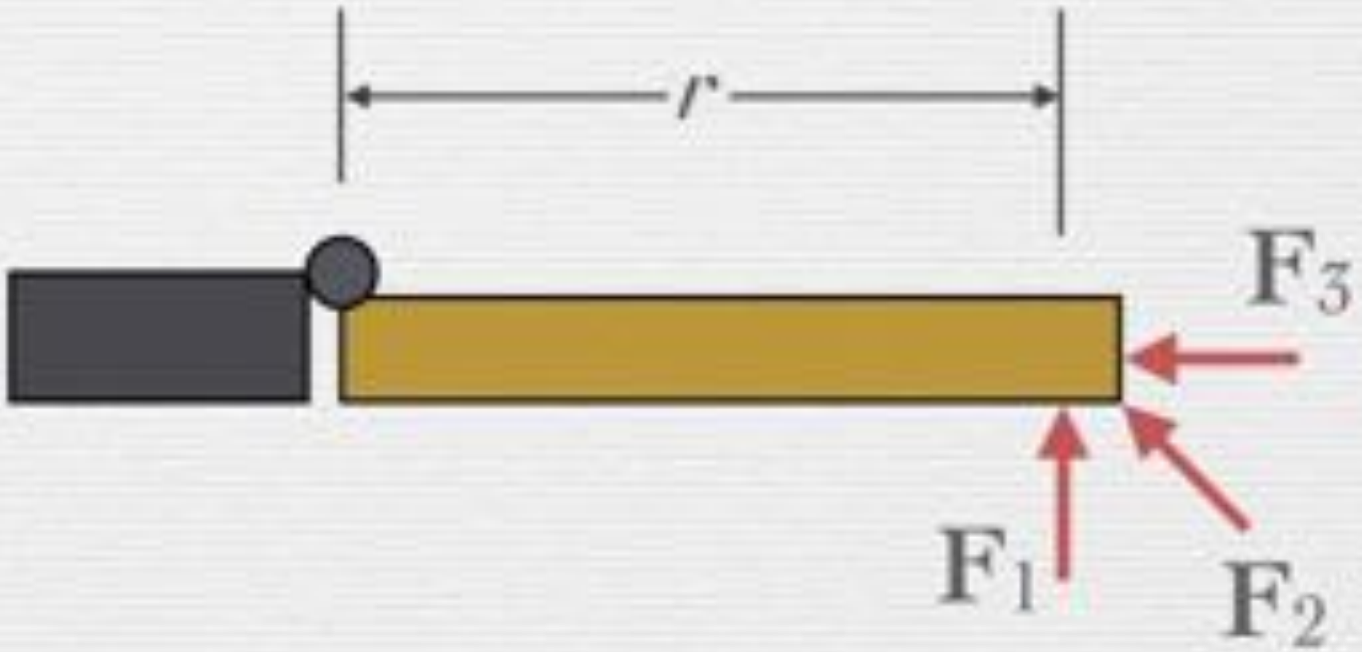


Torque (or why hobbit doors are a dumb design)

- The "twisting force" which cause rotation is called the **torque** (τ)
  - $\tau = r F_{\perp}$
  - Measure in Nm
- $\alpha \propto \tau$ (just like $a \propto F$)

Torque

- $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ might all be the same magnitude and the same distance r from the hinge
    - but they will *not* all result in the same twisting motion
    - only the *perpendicular* component of the force will contribute to rotation



Torque

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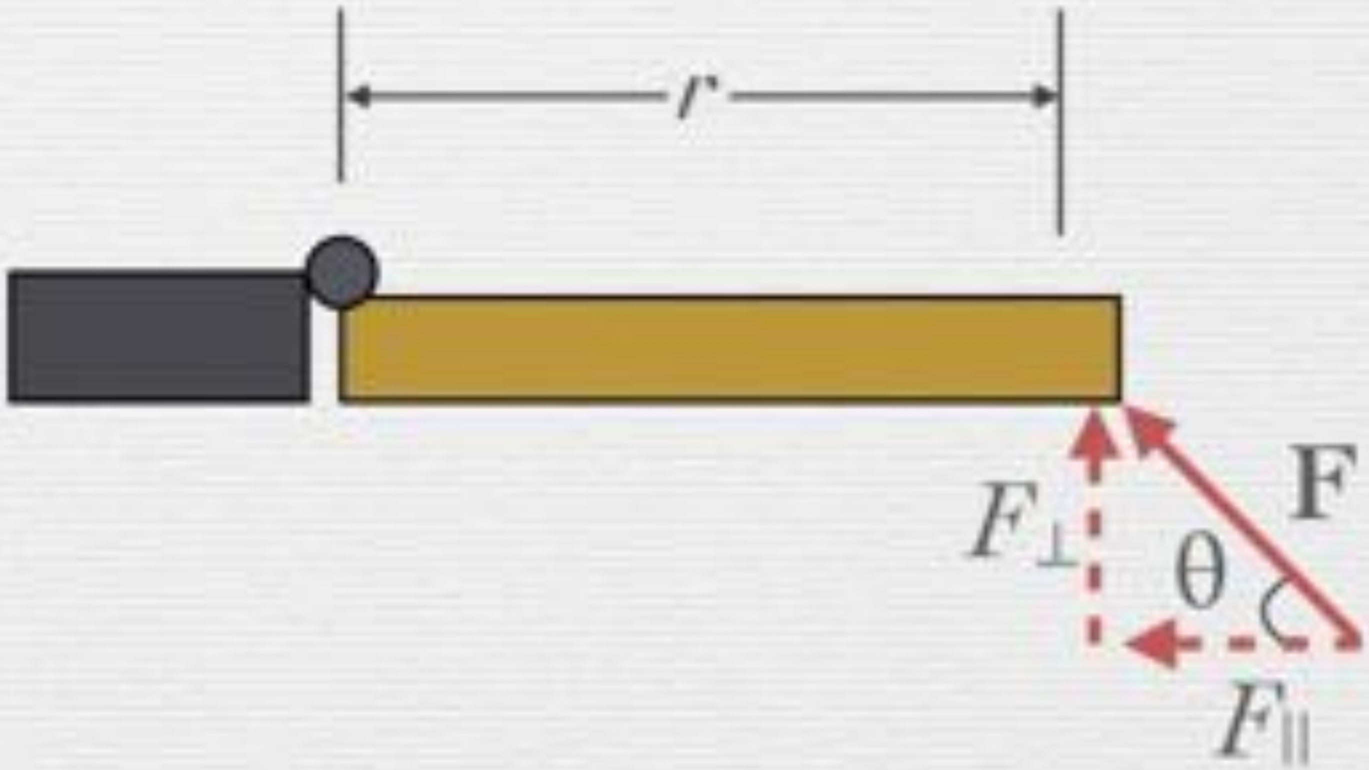
    - but they will *not* all result in the same twisting motion
    - only the *perpendicular* component of the force will contribute to rotation
- $\tau = r\,F\,\sin\theta$



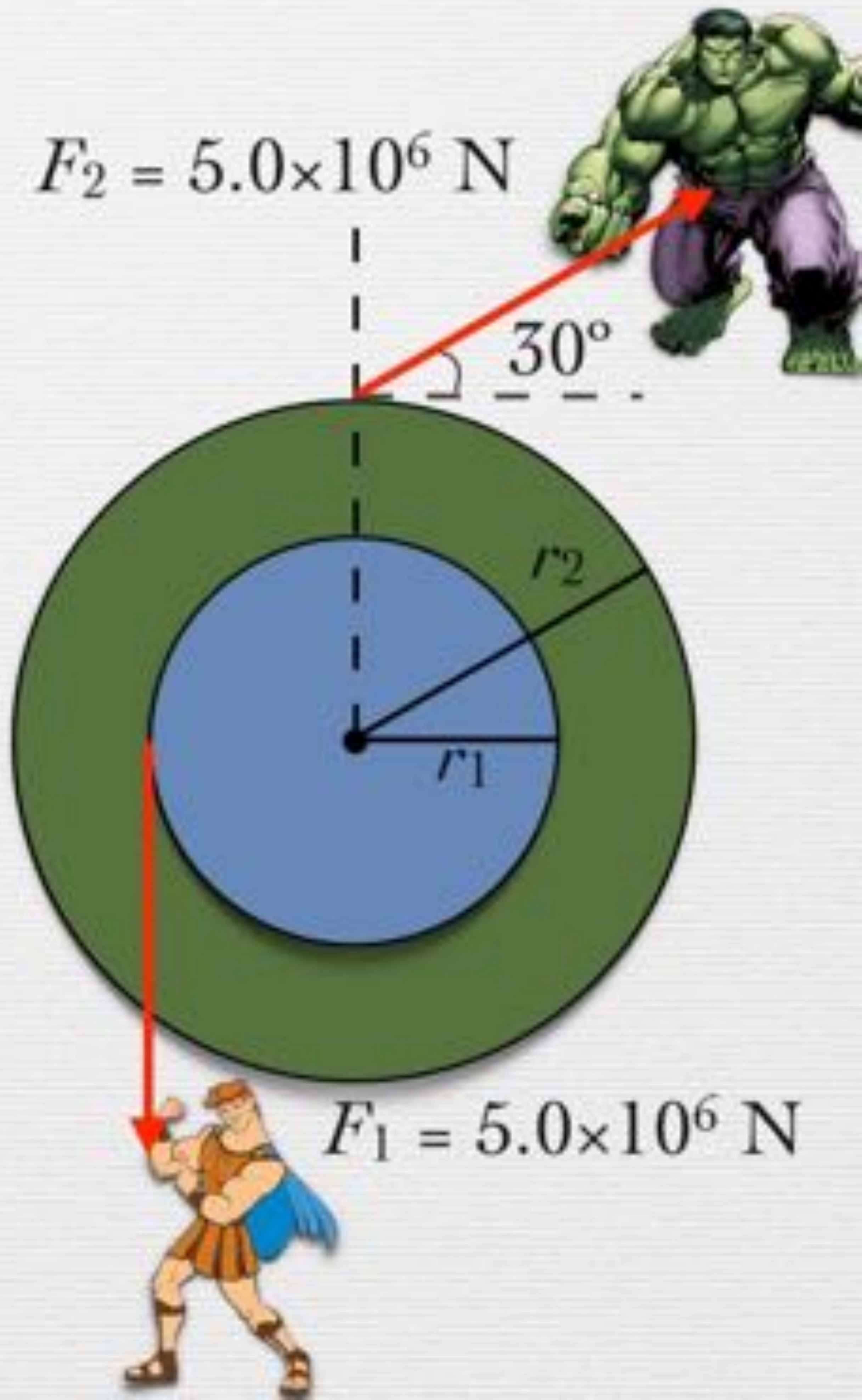
Example 5

- Hercules and the Hulk are in competition for some reason. They've matched each other in every test of strength, so Bruce Banner devises the following tug-of-war-esque challenge.

- Two thin cylindrical wheels, of radii r_1 = 3.0 m and r_2 = 5.0 m, are attached to each other on an axel that passes through the center of each.

- Both apply 5 million N of force as shown to the right. Who wins? What's the net torque?

    - *Ans. The Hulk wins.* $\tau_{net} = -6.7\times10^6$ *Nm*

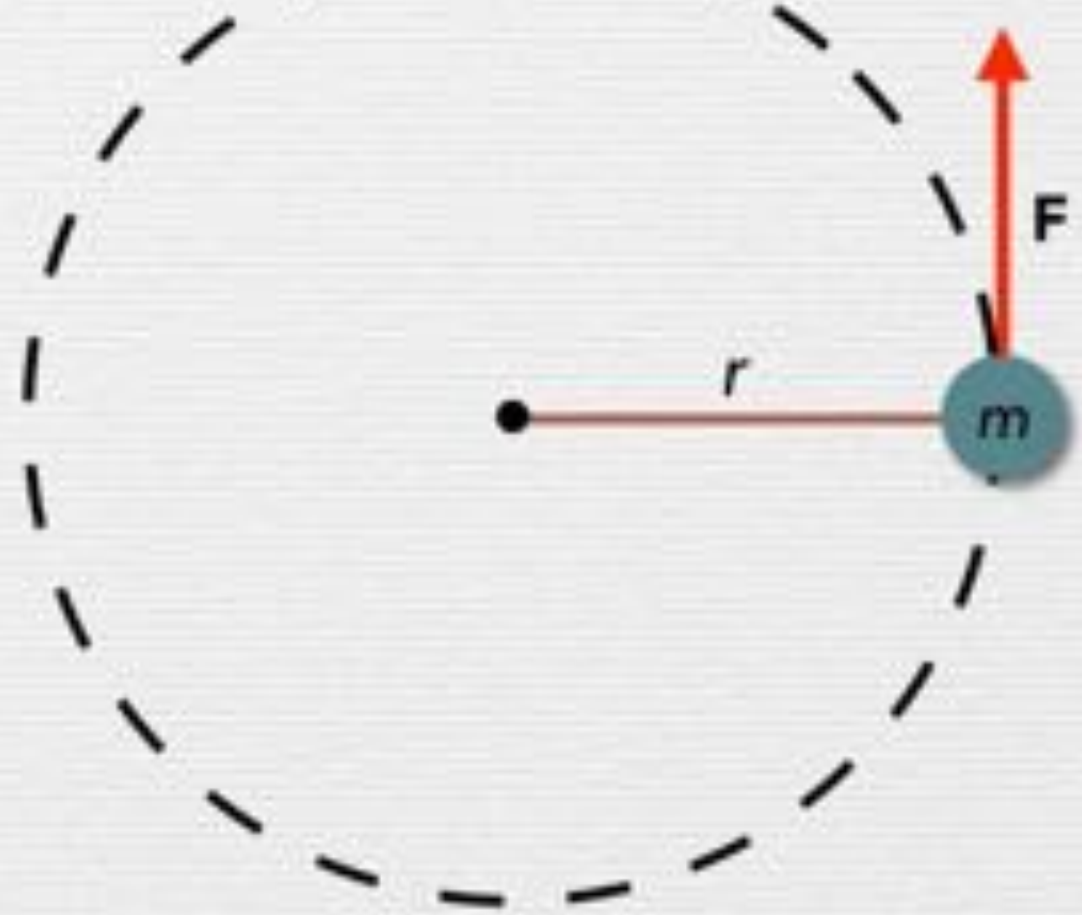


Rotational Inertia

- Linear acceleration
  - $a = \sum F / m$
- Angular acceleration
  - $\alpha = \sum \tau / I$
- Moment of inertia (or rotational inertia) is a measure of a body's resistance to changes in its rotation
  - Rotational "laziness"

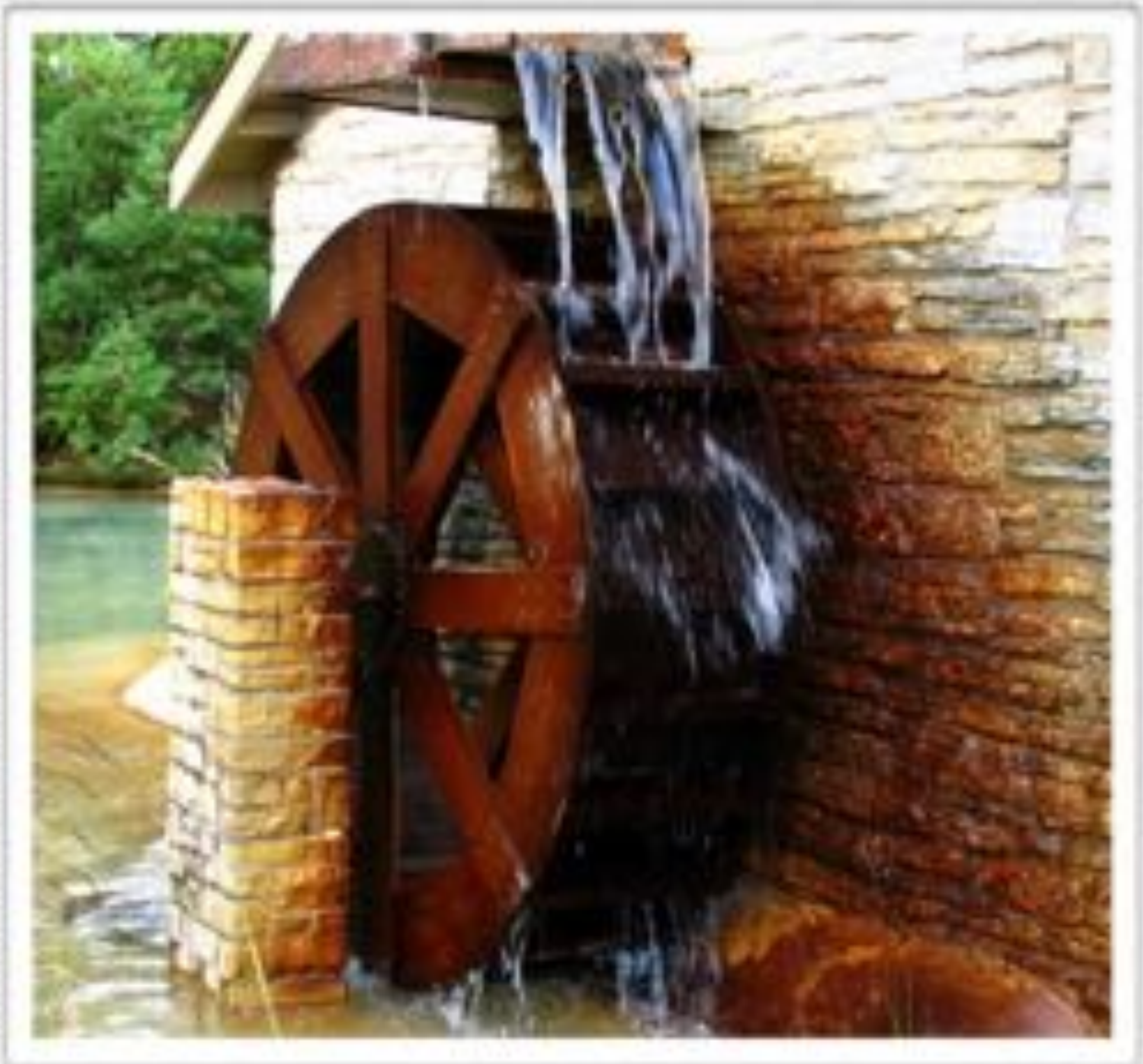
Rotational Inertia

- Picture a particle of mass m revolving in a circle of radius r
- Initially at rest, we want this particle to start rotating, so we give it a push
- $F = ma$
- $F = mr\alpha$
- $\tau = mr^2\alpha$
  - mr^2 represents the moment of inertia of the particle (measured in $kg \cdot m^2$)



Rotational Inertia

- Consider a rotating rigid body
  - basically a collection of particles all at varying distances from the axis of rotation
- $\sum\tau = \sum(mr^2)\alpha$
  - $I = \sum mr^2$
- $\sum\tau = I\alpha$
  - Newton's 2nd Law for rotation



Things to Note

- Moment of inertia (*I*) plays the same role for rotational motion that mass plays for translational motion
- The rotational inertia of an object depends not only on its mass, but also on how that mass is distributed with respect to the axis of rotation
- A large-diameter cylinder will have greater rotational inertia than a smaller-diameter cylinder of equal mass
  - The former will be harder to start rotating and harder to stop

Example 6

- Two weights of mass 5.0 kg and 7.0 kg are mounted 4.0 m apart on a massless rod. Calculate the moment of inertia of the system

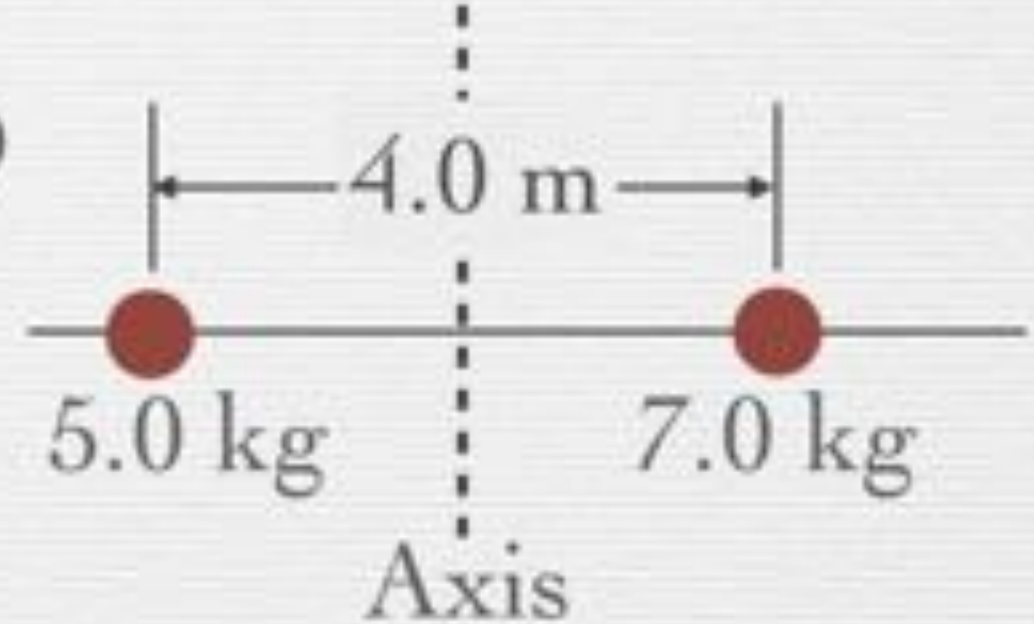
   a) when rotated about an axis halfway between the weights

   b) when the system rotates about an axis 0.50 m to the left of the 5.0-kg-mass

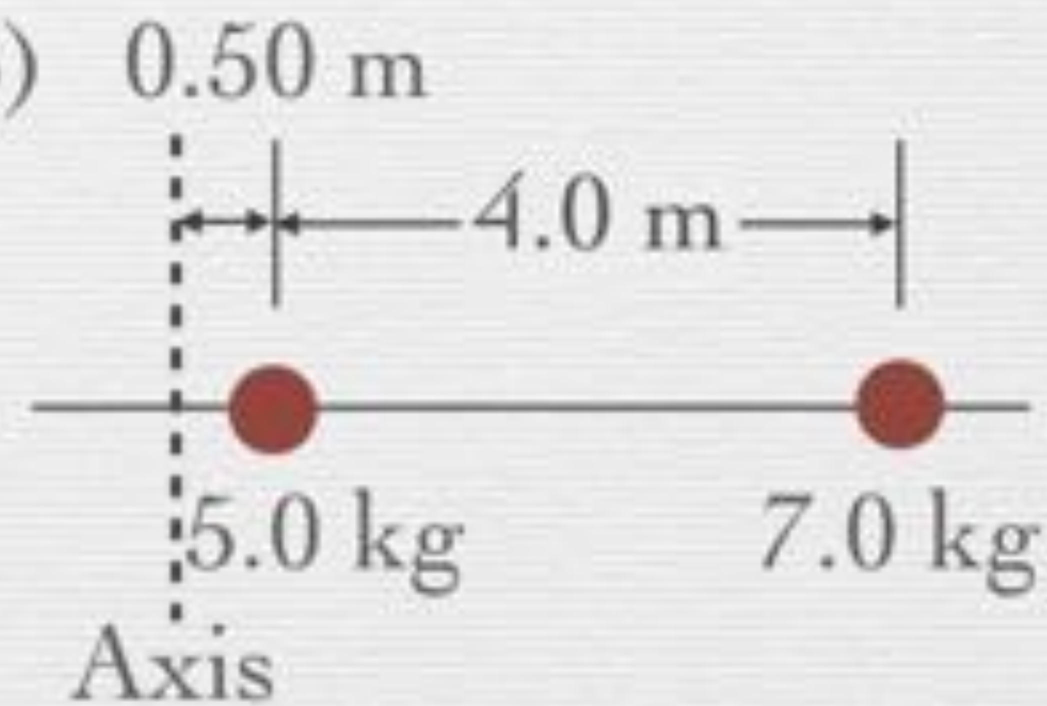
a) $I = 48\ kg \cdot m^2$

b) $I = 143\ kg \cdot m^2$

a)



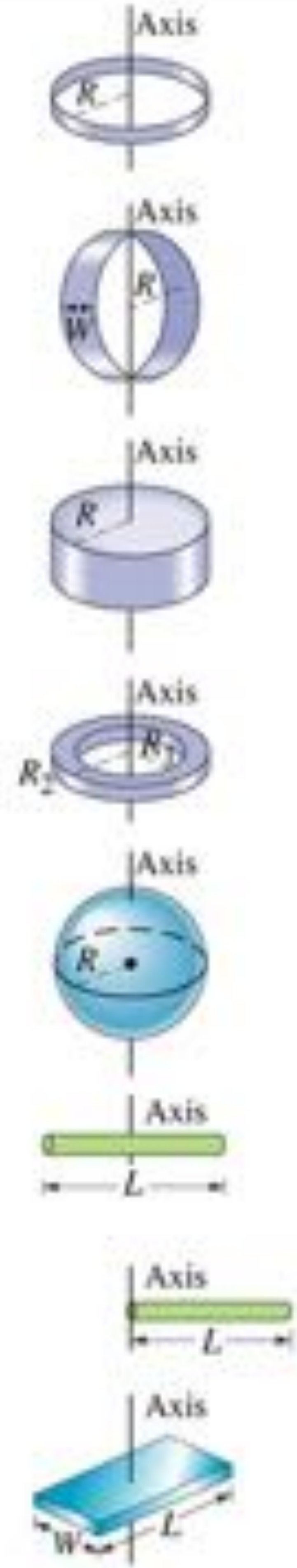
b)



Moment of Inertia

- Don’t memorize
- *Do*
    - roughly how they rank from greatest to least
    - what that implies about their behavior

	Object	Location of axis	Moment of inertia
(a)	**Thin hoop,** radius R	Through center	MR^2
(b)	**Thin hoop,** radius R width W	Through central diameter	$\frac{1}{2}MR^2 + \frac{1}{12}MW^2$
(c)	**Solid cylinder,** radius R	Through center	$\frac{1}{2}MR^2$
(d)	**Hollow cylinder,** inner radius R_1 outer radius R_2	Through center	$\frac{1}{2}M(R_1^2 + R_2^2)$
(e)	**Uniform sphere,** radius R	Through center	$\frac{2}{5}MR^2$
(f)	**Long uniform rod,** length L	Through center	$\frac{1}{12}ML^2$
(g)	**Long uniform rod,** length L	Through end	$\frac{1}{3}ML^2$
(h)	**Rectangular thin plate,** length L, width W	Through center	$\frac{1}{12}M(L^2 + W^2)$



Example 7

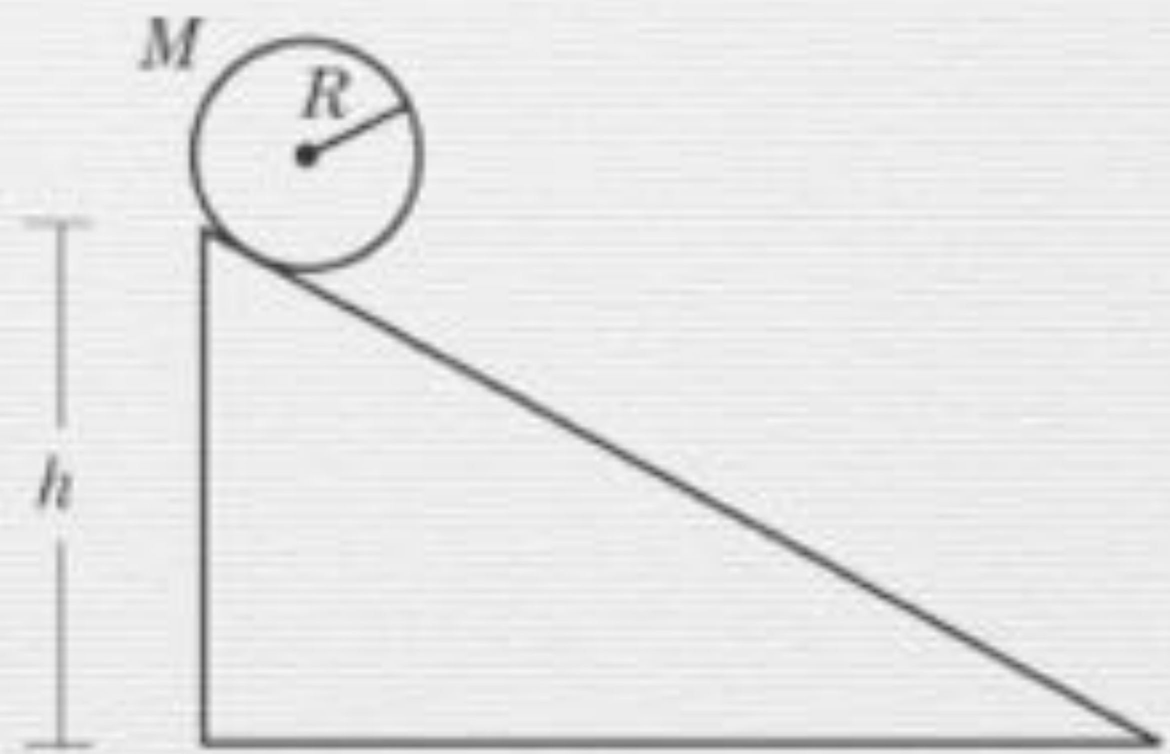
- A 15.0 N force is applied to a cord wrapped around a pulley of mass M = 4.00 kg and radius R = 33.0 cm. The pulley is observed to accelerate uniformly from rest to reach an angular speed of 30.0 rad/s in 3.00 s. If there is a frictional torque (at the axle), τ_{fr} = 1.10 Nm, determine the moment of inertia of the pulley.
    - $\sum \tau = 3.85\ Nm$
    - $\alpha = 10.0\ rad/s^2$
    - Ans. $I = 0.385\ kg \cdot m^2$

Rotational Kinetic Energy

- Translational kinetic energy
  - $KE_{tran} = ½\, mv^2$
- Rotational kinetic energy
  - $KE_{rot} = ½\, I\omega^2$
  - Make sure, do the units check out?
- $KE_{total} = KE_{tran} + KE_{rot}$

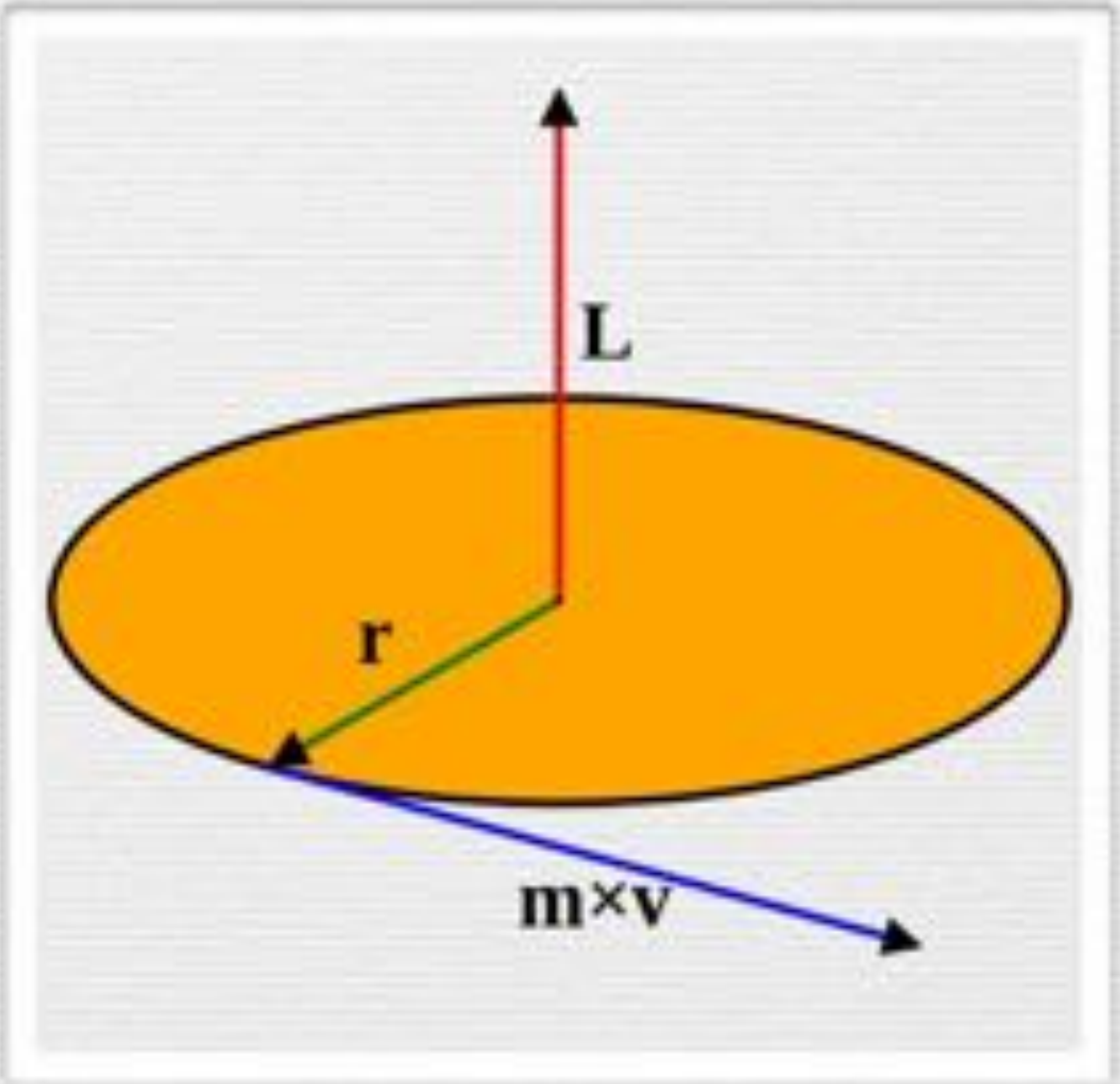
Example 8

- What will be the speed of a solid sphere of mass *M* and radius *R* when it reaches the bottom of an incline if it starts from rest at a vertical height *H* and rolls without slipping?
  - *Ans.* $v = \sqrt{\frac{10}{7} gH}$
- How does this compare with an object *sliding* down a frictionless incline.
  - *Ans.* $v = \sqrt{2gH}$



Angular Momentum

- Linear momentum
  - $p = mv$
- Angular momentum
  - $L = I\omega$
  - Measured in $kg \cdot m^2/s$
- Newton's 2nd Law (linear)
  - $\sum F = \Delta p / \Delta t$
- Newton's 2nd Law (angular)
  - $\sum \tau = \Delta L / \Delta t$

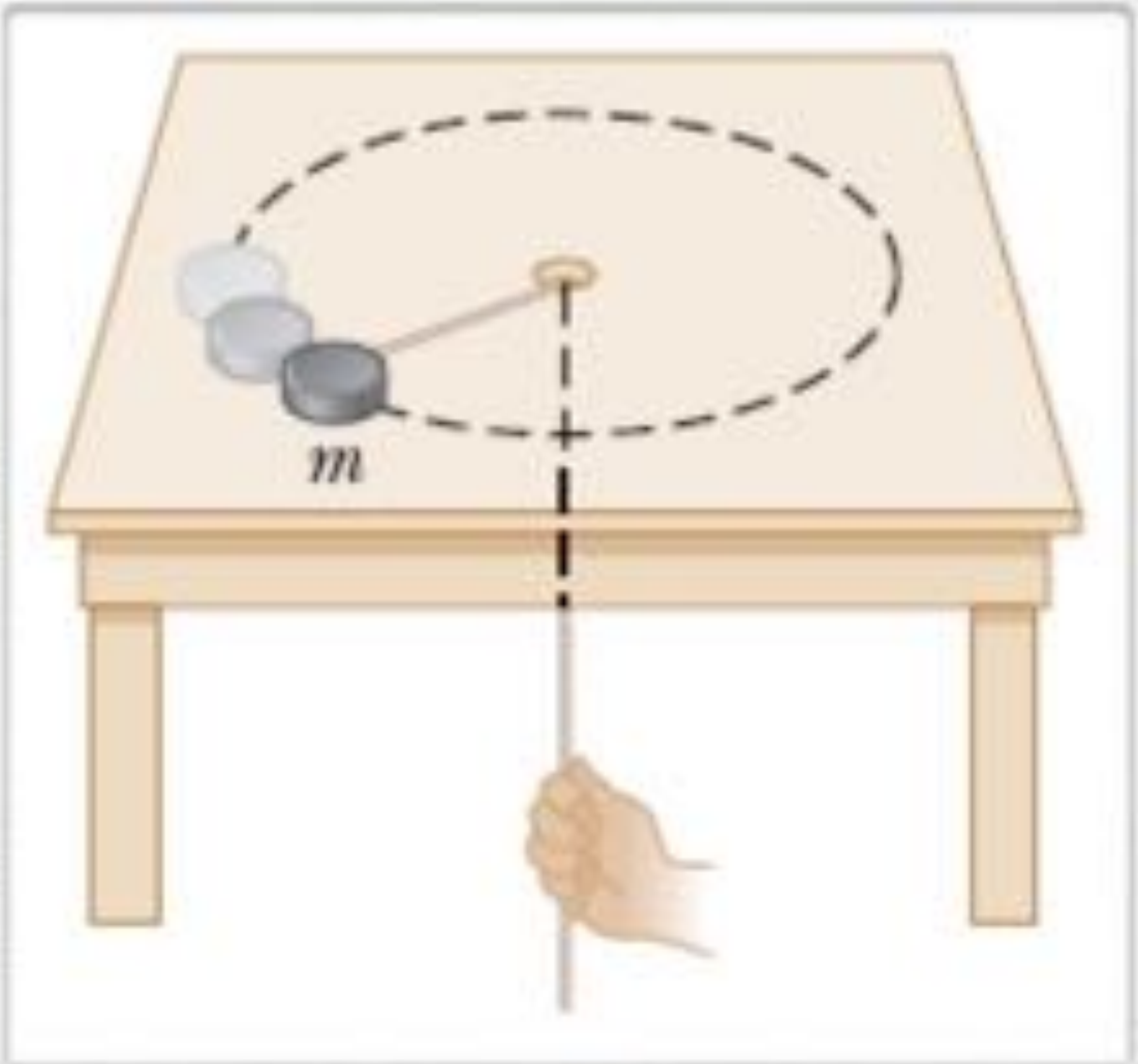


Conservation of Angular Momentum

- The total angular momentum of a rotating body remains constant if the net torque acting on it is zero

Example 9

- A mass m attached to the end of a string revolves in a circle on a frictionless tabletop. The other end of the string passes through a hole in the table. Initially, the mass revolves with a speed $v_1 = 2.4$ m/s in a circle of radius $r_1 = 0.80$ m. The string is then pulled slowly through the hole so that the radius is reduced to $r_2 = 0.48$ m. What is the speed, v_2, of the mass now?



  - *Ans.* $v_2 = 4.0$ *m/s*

- Why do you hold your arms out when trying to hold your balance (on a balance beam, tightrope, slack line, curb etc.)?
  - Holding your arms out increases your rotational inertial, making it harder for you to tip over.

- Why do figure skaters pull their arms and legs in when performing quick spins?
  - $L = I\omega$ is conserved
  - If you *decrease* your moment of inertia, you *increase* your angular speed