Section 4.6 Exercises

In Exercises 1–41, assume all variables are differentiable functions of t.

1. ***Area*** The radius r and area A of a circle are related by the equation $A = \pi r^2$. Write an equation that relates dA/dt to dr/dt.

2. ***Surface Area*** The radius r and surface area S of a sphere are related by the equation $S = 4\pi r^2$. Write an equation that relates dS/dt to dr/dt.

3. ***Volume*** The radius r, height h, and volume V of a right circular cylinder are related by the equation $V = \pi r^2 h$.

   (a) How is dV/dt related to dh/dt if r is constant?

   (b) How is dV/dt related to dr/dt if h is constant?

   (c) How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

4. ***Electrical Power*** The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.

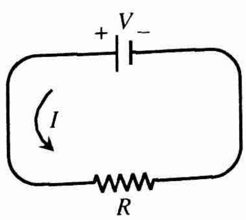
   (a) How is dP/dt related to dR/dt and dI/dt?

   (b) How is dR/dt related to dI/dt if P is constant?

5. ***Diagonals*** If x, y, and z are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$. How is ds/dt related to dx/dt, dy/dt, and dz/dt?

6. ***Area*** If a and b are the lengths of two sides of a triangle, and θ the measure of the included angle, the area A of the triangle is $A = (1/2)\, ab \sin\theta$. How is dA/dt related to da/dt, db/dt, and $d\theta/dt$?

7. ***Changing Voltage*** The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of 1/3 amp/sec. Let t denote time in sec.



   (a) What is the value of dV/dt?

   (b) What is the value of dI/dt?

   (c) Write an equation that relates dR/dt to dV/dt and dI/dt.

   (d) Writing to Learn Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing, or decreasing? Explain.

8. ***Heating a Plate*** When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/sec. At what rate is the plate's area increasing when the radius is 50 cm?

9. ***Changing Dimensions in a Rectangle*** The length ℓ of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $\ell = 12$ cm and $w = 5$ cm, find the rates of change of

   (a) the area, **(b)** the perimeter, and

   (c) the length of a diagonal of the rectangle.

   (d) Writing to Learn Which of these quantities are decreasing, and which are increasing? Explain.

10. ***Changing Dimensions in a Rectangular Box*** Suppose that the edge lengths x, y, and z of a closed rectangular box are changing at the following rates:

$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$

   Find the rates at which the box's **(a)** volume, **(b)** surface area, and **(c)** diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

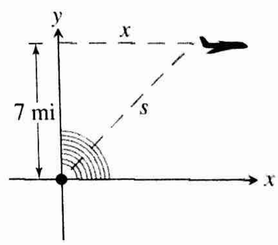
11. ***Inflating Balloon*** A spherical balloon is inflated with helium at the rate of 100π ft³/min.

    (a) How fast is the balloon's radius increasing at the instant the radius is 5 ft?

    (b) How fast is the surface area increasing at that instant?

12. ***Growing Raindrop*** Suppose that a droplet of mist is a perfect sphere and that, through condensation, the droplet picks up moisture at a rate proportional to its surface area. Show that under these circumstances the droplet's radius increases at a constant rate.

13. ***Air Traffic Control*** An airplane is flying at an altitude of 7 mi and passes directly over a radar antenna as shown in the figure. When the plane is 10 mi from the antenna ($s = 10$), the radar detects that the distance s is changing at the rate of 300 mph. What is the speed of the airplane at that moment?



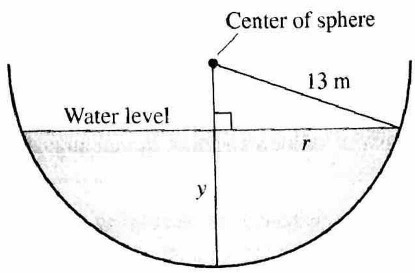
14. ***Flying a Kite*** Inge flies a kite at a height of 300 ft, the wind carrying the kite horizontally away at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?

15. ***Boring a Cylinder*** The mechanics at Lincoln Automotive are reboring a 6-in. deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.?

16. ***Growing Sand Pile*** Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high? Give your answer in cm/min.

17. ***Draining Conical Reservoir*** Water is flowing at the rate of 50 m^3/min from a concrete conical reservoir (vertex down) of base radius 45 m and height 6 m. **(a)** How fast is the water level falling when the water is 5 m deep? **(b)** How fast is the radius of the water's surface changing at that moment? Give your answer in cm/min.

18. ***Draining Hemispherical Reservoir*** Water is flowing at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y units deep.

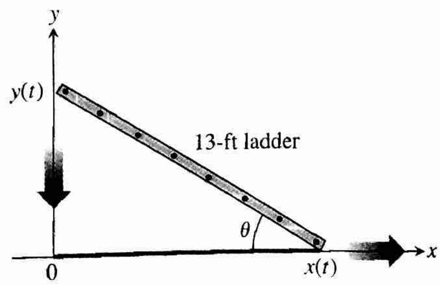


(a) At what rate is the water level changing when the water is 8 m deep?

(b) What is the radius r of the water's surface when the water is y m deep?

(c) At what rate is the radius r changing when the water is 8 m deep?

19. ***Sliding Ladder*** A 13-ft ladder is leaning against a house (see figure) when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

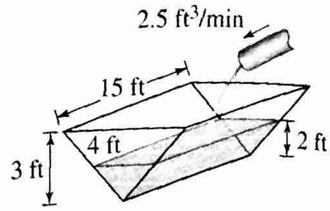


(a) How fast is the top of the ladder sliding down the wall at that moment?

(b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing at that moment?

(c) At what rate is the angle θ between the ladder and the ground changing at that moment?

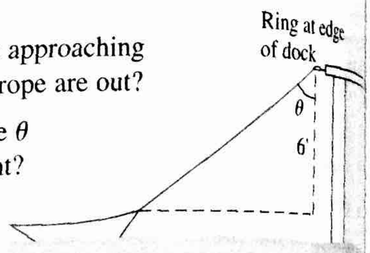
20. ***Filling a Trough*** A trough is 15 ft long and 4 ft across the top as shown in the figure. Its ends are isosceles triangles with height 3 ft. Water runs into the trough at the rate of 2.5 ft^3/min. How fast is the water level rising when it is 2 ft deep?



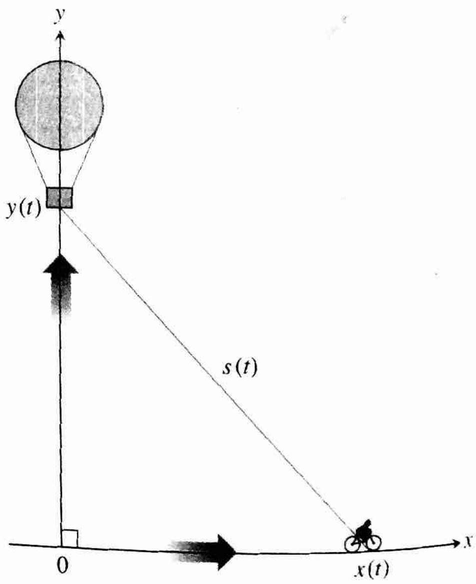
21. ***Hauling in a Dinghy*** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow as shown in the figure. The rope is hauled in at the rate of 2 ft/sec.

(a) How fast is the boat approaching the dock when 10 ft of rope are out?

(b) At what rate is angle θ changing at that moment?



22. ***Rising Balloon*** A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance between the bicycle and balloon increasing 3 sec later (see figure)?



In Exercises 23 and 24, a particle is moving along the curve $y = f(x)$.

23. Let $y = f(x) = \dfrac{10}{1 + x^2}$.

If $dx/dt = 3$ cm/sec, find dy/dt at the point where

(a) $x = -2$. **(b)** $x = 0$. **(c)** $x = 20$.

24. Let $y = f(x) = x^3 - 4x$.

If $dx/dt = -2$ cm/sec, find dy/dt at the point where

(a) $x = -3$. **(b)** $x = 1$. **(c)** $x = 4$.

25. ***Particle Motion*** A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (in meters) increases at a constant rate of 10 m/sec. How fast is the angle of inclination θ of theline joining the particle to the origin changing when $x = 3$?

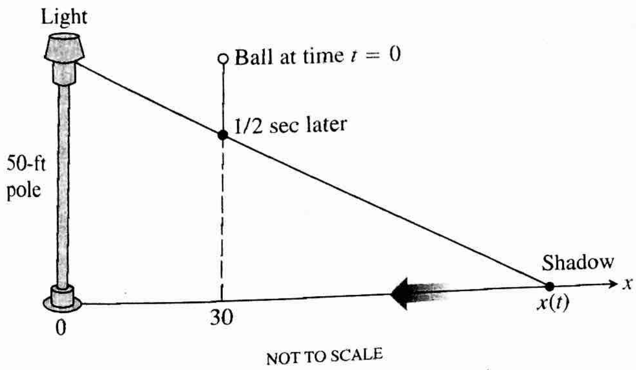
26. ***Particle Motion*** A particle moves from right to left along the parabolic curve $y = \sqrt{-x}$ in such a way that its x-coordinate (in meters) decreases at the rate of 8 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = -4$?

27. ***Melting Ice*** A spherical iron ball is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 8 mL/min, how fast is the outer surface area of ice decreasing when the outer diameter (ball plus ice) is 20 cm?

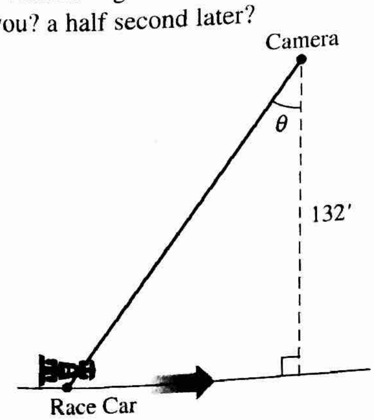
28. ***Particle Motion*** A particle $P(x, y)$ is moving in the coordinate plane in such a way that $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

29. ***Moving Shadow*** A man 6 ft tall walks at the rate of 5 ft/sec toward a streetlight that is 16 ft above the ground. At what rate is the length of his shadow changing when he is 10 ft from the base of the light?

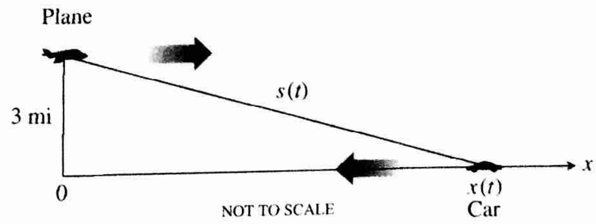
30. ***Moving Shadow*** A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light as shown below. How fast is the ball's shadow moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ in t sec.)



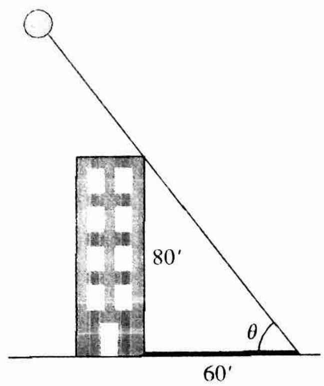
31. ***Moving Race Car*** You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mph (264 ft/sec) as shown in the figure. About how fast will your camera angle θ be changing when the car is right in front of you? a half second later?



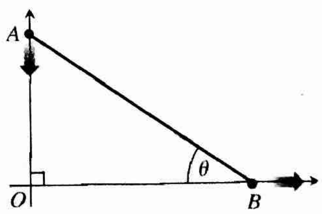
32. ***Speed Trap*** A highway patrol airplane flies 3 mi above a level, straight road at a constant rate of 120 mph. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi the line-of-sight distance is decreasing at the rate of 160 mph. Find the car's speed along the highway.



33. ***Building's Shadow*** On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long as shown in the figure. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of 0.27°/min. At what rate is the shadow length decreasing? Express your answer in in./min, to the nearest tenth. (Remember to use radians.)



34. ***Walkers*** A and B are walking on straight streets that meet at right angles. A approaches the intersection at 2 m/sec and B moves away from the intersection at 1 m/sec as shown in the figure. At what rate is the angle θ changing when A is 10 m from the intersection and B is 20 m from the intersection? Express your answer in degrees per second to the nearest degree.



35. ***Moving Ships*** Two ships are steaming away from a point O along routes that make a 120° angle. Ship A moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yards). Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

36. True or False If the radius of a circle is expanding at a constant rate, then its circumference is increasing at a constant rate. Justify your answer.

37. True or False If the radius of a circle is expanding at a constant rate, then its area is increasing at a constant rate. Justify your answer.

38. Multiple Choice If the volume of a cube is increasing at 24 in^3/min and each edge of the cube is increasing at 2 in./min, what is the length of each edge of the cube?

(A) 2 in. (B) $2\sqrt{2}$ in. (C) $\sqrt[3]{12}$ in. (D) 4 in. (E) 8 in.

39. Multiple Choice If the volume of a cube is increasing at 24 in^3/min and the surface area of the cube is increasing at 12 in^2/min, what is the length of each edge of the cube?

(A) 2 in. (B) $2\sqrt{2}$ in. (C) $\sqrt[3]{12}$ in. (D) 4 in. (E) 8 in.

40. Multiple Choice A particle is moving around the unit circle (the circle of radius 1 centered at the origin). At the point (0.6, 0.8) the particle has horizontal velocity $dx/dt = 3$. What is its vertical velocity dy/dt at that point?

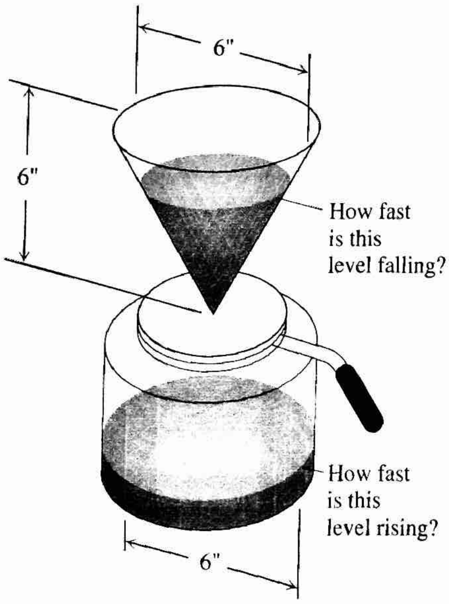
(A) −3.875 (B) −3.75 (C) −2.25 (D) 3.75 (E) 3.875

41. Multiple Choice A cylindrical rubber cord is stretched at a constant rate of 2 cm per second. Assuming its volume does not change, how fast is its radius shrinking when its length is 100 cm and its radius is 1 cm?

(A) 0 cm/sec (B) 0.01 cm/sec (C) 0.02 cm/sec
(D) 2 cm/sec (E) 3.979 cm/sec

Explorations

42. *Making Coffee* Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min.



(a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

(b) How fast is the level in the cone falling at that moment?

43. *Cost, Revenue, and Profit* A company can manufacture x items at a cost of $c(x)$ dollars, a sales revenue of $r(x)$ dollars, and a profit of $p(x) = r(x) - c(x)$ dollars (all amounts in thousands). Find dc/dt, dr/dt, and dp/dt for the following values of x and dx/dt.

(a) $r(x) = 9x$, $c(x) = x^3 - 6x^2 + 15x$, and $dx/dt = 0.1$ when $x = 2$.

(b) $r(x) = 70x$, $c(x) = x^3 - 6x^2 + 45/x$, and $dx/dt = 0.05$ when $x = 1.5$.

44. Group Activity *Cardiac Output* In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würtzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 liters a minute. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where Q is the number of milliliters of CO_2 you exhale in a minute and D is the difference between the CO_2 concentration (mL/L) in the blood pumped to the lungs and the CO_2 concentration in the blood returning from the lungs. With $Q = 233$ mL/min and $D = 97 - 56 = 41$ mL/L,

$$y = \frac{233 \text{ mL/min}}{41 \text{ mL/L}} \approx 5.68 \text{ L/min},$$

fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. What is happening to the cardiac output?

Extending the Ideas

45. *Motion along a Circle* A wheel of radius 2 ft makes 8 revolutions about its center every second.

(a) Explain how the parametric equations

$$x = 2\cos\theta, \quad y = 2\sin\theta$$

can be used to represent the motion of the wheel.

(b) Express θ as a function of time t.

(c) Find the rate of horizontal movement and the rate of vertical movement of a point on the edge of the wheel when it is at the position given by $\theta = \pi/4$, $\pi/2$, and π.

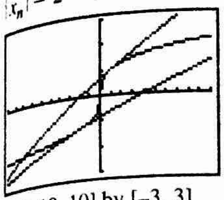
46. *Ferris Wheel* A Ferris wheel with radius 30 ft makes one revolution every 10 sec.

(a) Assume that the center of the Ferris wheel is located at the point (0, 40), and write parametric equations to model its motion. [*Hint:* See Exercise 45.]

(b) At $t = 0$ the point P on the Ferris wheel is located at (30, 40). Find the rate of horizontal movement, and the rate of vertical movement of the point P when $t = 5$ sec and $t = 8$ sec.

63. If $f'(x_1) \neq 0$, then x_2 and all later approximations are equal to x_1.

65. $x_2 = -2, x_3 = 4, x_4 = -8$, and $x_5 = 16$; $|x_n| = 2^{n-1}$.



[−10, 10] by [−3, 3]

67. Finding a zero of sin x by Newton's method would use the recursive formula $x_{n+1} = x_n - \frac{\sin(x_n)}{\cos(x_n)} = x_n - \tan x_n$, and that is exactly what the calculator would be doing. Any zero of sin x would be a multiple of π.

69. $$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x/\cos x}{x}$$
$$= \lim_{x\to 0}\left(\frac{1}{\cos x}\cdot\frac{\sin x}{x}\right)$$
$$= \left(\lim_{x\to 0}\frac{1}{\cos x}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right)$$
$$=(1)(1) = 1.$$

71. The linearization is $1 + \frac{3x}{2}$. It is the sum of the two individual linearizations.

Section 4.6

Quick Review 4.6

1. $\sqrt{74}$ 3. $\frac{1-2y}{2x+2y-1}$ 5. $2x\cos^2 y$

7. One possible answer: $x = -2 + 6t,\ y = 1 - 4t,\ 0 \le t \le 1$.

9. One possible answer: $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$

Exercises 4.6

1. $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

3. **(a)** $\frac{dV}{dt} = \pi r^2\frac{dh}{dt}$ **(b)** $\frac{dV}{dt} = 2\pi rh\frac{dr}{dt}$ **(c)** $\frac{dV}{dt} = \pi r^2\frac{dh}{dt} + 2\pi rh\frac{dr}{dt}$

5. $\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}}{\sqrt{x^2+y^2+z^2}}$

7. **(a)** 1 volt/sec **(b)** $-\frac{1}{3}$ amp/sec **(c)** $\frac{dV}{dt} = I\frac{dR}{dt} + R\frac{dI}{dt}$
(d) $\frac{dR}{dt} = \frac{3}{2}$ ohms/sec. R is increasing since $\frac{dR}{dt}$ is positive.

9. **(a)** $\frac{dA}{dt} = 14\ \text{cm}^2/\text{sec}$ **(b)** $\frac{dP}{dt} = 0$ cm/sec **(c)** $\frac{dD}{dt} = -\frac{14}{13}$ cm/sec
(d) The area is increasing, because its derivative is positive.
The perimeter is not changing, because its derivative is zero.
The diagonal length is decreasing, because its derivative is negative.

11. **(a)** 1 ft/min **(b)** $40\pi\ \text{ft}^2/\text{min}$ 13. $\frac{dx}{dt} = \frac{3000}{\sqrt{51}}$ mph ≈ 420.08 mph

15. $\frac{19\pi}{2500} \approx 0.0239\ \text{in}^3/\text{min}$

17. **(a)** $\frac{32}{9\pi} \approx 1.13$ cm/min **(b)** $-\frac{80}{3\pi} \approx -8.49$ cm/min

19. **(a)** 12 ft/sec **(b)** $-\frac{119}{2}\ \text{ft}^2/\text{sec}$ **(c)** −1 radian/sec

21. **(a)** $\frac{5}{2}$ ft/sec **(b)** $-\frac{3}{20}$ radian/sec

23. **(a)** $\frac{24}{5}$ cm/sec **(b)** 0 cm/sec **(c)** $-\frac{1200}{160{,}801} \approx -0.00746$ cm/sec

25. 1 radian/sec 27. $1.6\ \text{cm}^2/\text{min}$ 29. −3 ft/sec

31. In front: 2 radians/sec;
Half second later: 1 radian/sec

33. 7.1 in./min 35. 29.5 knots

37. **False.** Since $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$, the value of $\frac{dA}{dt}$ depends on r.

39. E 41. B

43. **(a)** $\frac{dc}{dt} = 0.3$ $\frac{dr}{dt} = 0.9$ $\frac{dp}{dt} = 0.6$
(b) $\frac{dc}{dt} = 1.5625$ $\frac{dr}{dt} = 3.5$ $\frac{dp}{dt} = 5.0625$

45. **(a)** The point being plotted would correspond to a point on the edge of the wheel as the wheel turns.
(b) One possible answer:
$\theta = 16\pi t$, where t is in seconds.
(c) Assuming counterclockwise motion, the rates are as follows.
$\theta = \frac{\pi}{4}$: $\frac{dx}{dt} \approx -71.086$ ft/sec
$\frac{dy}{dt} \approx 71.086$ ft/sec
$\theta = \frac{\pi}{2}$: $\frac{dx}{dt} \approx -100.531$ ft/sec
$\frac{dy}{dt} = 0$ ft/sec
$\theta = \pi$: $\frac{dx}{dt} = 0$ ft/sec
$\frac{dy}{dt} \approx -100.531$ ft/sec

47. **(a)** 9% per year
(b) Increasing at 1% per year

Quick Quiz (Sections 4.4–4.6)

1. B 3. A

Chapter 4 Review Exercises

1. Maximum: $\frac{4\sqrt{6}}{9}$ at $x = \frac{4}{3}$;
minimum: −4 at $x = -2$

2. No global extrema

3. **(a)** $[-1, 0)$ and $[1, \infty)$ **(b)** $(-\infty, -1]$ and $(0, 1]$ **(c)** $(-\infty, 0)$ and $(0, \infty)$
(d) None **(e)** Local minima at $(1, e)$ and $(-1, e)$ **(f)** None

4. **(a)** $[-\sqrt{2}, \sqrt{2}]$ **(b)** $[-2, -\sqrt{2}]$ and $[\sqrt{2}, 2]$ **(c)** $(-2, 0)$ **(d)** $(0, 2)$
(e) Local max: $(-2, 0)$ and $(\sqrt{2}, 2)$; local min: $(2, 0)$ and $(-\sqrt{2}, -2)$
(f) $(0, 0)$

5. **(a)** Approximately $(-\infty, 0.385]$ **(b)** Approximately $[0.385, \infty)$
(c) None **(d)** $(-\infty, \infty)$ **(e)** Local maximum at $\approx (0.385, 1.215)$ **(f)** None

6. **(a)** $[1, \infty)$ **(b)** $(-\infty, 1]$ **(c)** $(-\infty, \infty)$ **(d)** None
(e) Local minimum at $(1, 0)$ **(f)** None

7. **(a)** $[0, 1)$ **(b)** $(-1, 0]$ **(c)** $(-1, 1)$ **(d)** None
(e) Local minimum at $(0, 1)$ **(f)** None

8. **(a)** $(-\infty, -2^{-1/3}] \approx (-\infty, -0.794]$
(b) $[-2^{-1/3}, 1) \approx [-0.794, 1)$ and $(1, \infty)$
(c) $(-\infty, -2^{1/3}) \approx (-\infty, -1.260)$ and $(1, \infty)$ **(d)** $(-1.260, 1)$