10.3 Polar Functions

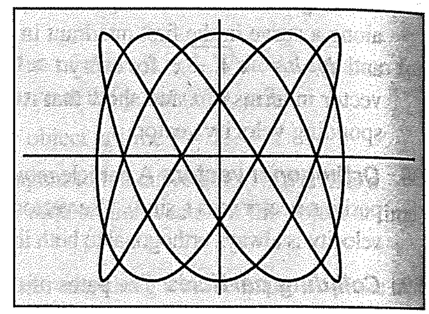
What you'll learn about

- Polar Coordinates
- Polar Curves
- Slopes of Polar Curves
- Areas Enclosed by Polar Curves
- A Small Polar Gallery

. . . and why

Polar equations enable us to define some interesting and important curves that would be difficult or impossible to define in the form $y = f(x)$.

Polar Coordinates



If you graph the two functions $y = \sin 3x$ and $y = \cos 5x$ on the same pair of axes, you will get two sinusoids. But if you graph the curve defined *parametrically* by $x = \sin 3t$ and $y = \cos 5t$, you will get the figure shown. Parametric graphing opens up a whole new world of curves that can be defined using our familiar basic functions.

Another way to enter that world is to use a different coordinate system. In **polar coordinates** we identify the origin O as the **pole** and the positive x-axis as the **initial ray** of angles measured in the usual trigonometric way. We can then identify each point P in the plane by polar coordinates (r, θ), where r gives the directed distance from O to P and θ gives the directed angle from the initial ray to the ray $\overrightarrow{OP}$. In Figure 10.19 we see that the point P with rectangular (Cartesian) coordinates (2, 2) has polar coordinates $(2\sqrt{2}, \pi/4)$.

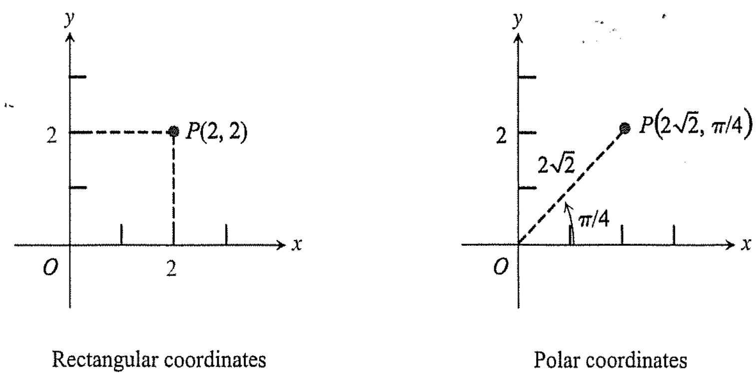


Figure 10.19 Point P has rectangular coordinates (2, 2) and polar coordinates $(2\sqrt{2}, \pi/4)$.

As you would expect, we can also coordinatize point P with the polar coordinates $(2\sqrt{2}, 9\pi/4)$ or $(2\sqrt{2}, -7\pi/4)$, since those angles determine the same ray $\overrightarrow{OP}$. Less obviously, we can also coordinatize P with polar coordinates $(-2\sqrt{2}, -3\pi/4)$, since the *directed* distance $-2\sqrt{2}$ in the $-3\pi/4$ direction is the same as the directed distance $2\sqrt{2}$ in the $\pi/4$ direction (Figure 10.20). So, although each pair (r, θ) determines a unique point in the plane, each point in the plane can be coordinatized by an infinite number of polar ordered pairs.

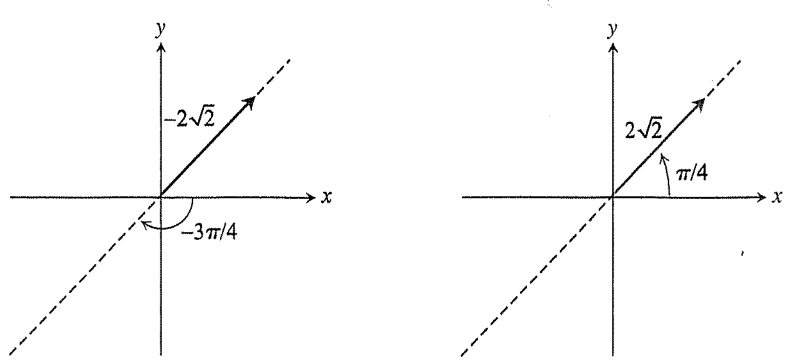


Figure 10.20 The directed *negative* distance $-2\sqrt{2}$ in the $-3\pi/4$ direction is the same as the directed *positive* distance $2\sqrt{2}$ in the $\pi/4$ direction. Thus the polar coordinates $(-2\sqrt{2}, -3\pi/4)$ and $(2\sqrt{2}, \pi/4)$ determine the same point.

EXAMPLE 1 Rectangular and Polar Coordinates

(a) Find rectangular coordinates for the points with given polar coordinates.

(i) $(4, \pi/2)$ **(ii)** $(-3, \pi)$ **(iii)** $(16, 5\pi/6)$ **(iv)** $(-\sqrt{2}, -\pi/4)$

(b) Find two different sets of polar coordinates for the points with given rectangular coordinates.

(i) $(1, 0)$ **(ii)** $(-3, 3)$ **(iii)** $(0, -4)$ **(iv)** $(1, \sqrt{3})$

SOLUTION

(a) (i) $(0, 4)$ **(ii)** $(3, 0)$ **(iii)** $(-8\sqrt{3}, 8)$ **(iv)** $(-1, 1)$

(b) A point has infinitely many sets of polar coordinates, so here we list just two typical examples for each given point.

(i) $(1, 0), (1, 2\pi)$ **(ii)** $(3\sqrt{2}, 3\pi/4), (-3\sqrt{2}, -\pi/4)$

(iii) $(4, -\pi/2), (4, 3\pi/2)$ **(iv)** $(2, \pi/3), (-2, 4\pi/3)$

Now try Exercises 1 and 3.

EXAMPLE 2 Graphing with Polar Coordinates

Graph all points in the plane that satisfy the given polar equation

(a) $r = 2$ **(b)** $r = -2$ **(c)** $\theta = \pi/6$

SOLUTION

First, note that we do *not* label our axes r and θ. We are graphing *polar* equations in the usual xy-plane, not renaming our rectangular variables!

(a) The set of all points with directed distance 2 units from the pole is a circle of radius 2 centered at the origin (Figure 10.21a).

(b) The set of all points with directed distance -2 units from the pole is also a circle of radius 2 centered at the origin (Figure 10.21b).

(c) The set of all points of positive or negative directed distance from the pole in the $\pi/6$ direction is a line through the origin with slope $\tan(\pi/6)$ (Figure 10.21c).

Now try Exercise 7.

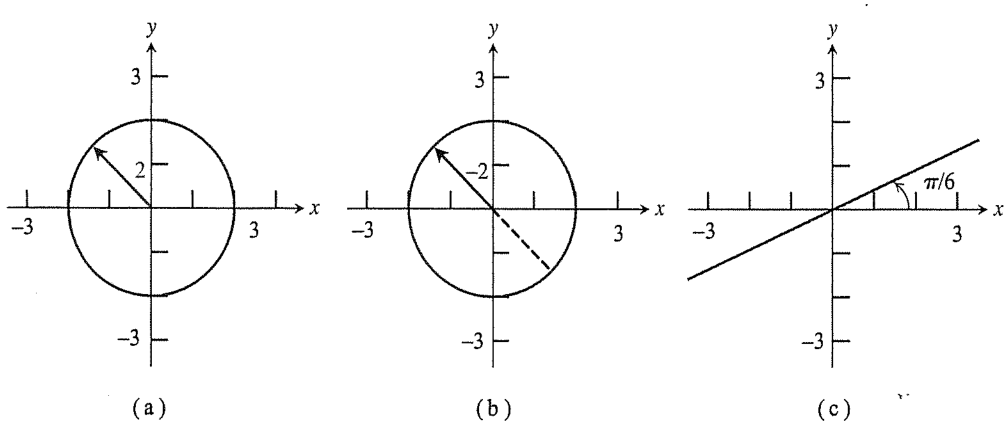


Figure 10.21 Polar graphs of (a) $r = 2$, (b) $r = -2$, and (c) $\theta = \pi/6$. (Example 2)

Polar Curves

The curves in Example 2 are a start, but we would not introduce a new coordinate system just to graph circles and lines; there are far more interesting polar curves to study. In the past it was hard work to produce reasonable polar graphs by hand, but today, thanks to graphing technology, it is just a matter of finding the right window and pushing the right buttons. Our intent in this section is to use the technology to produce the graphs and then concentrate on how calculus can be used to give us further information.

EXAMPLE 3 Polar Graphing with Technology

Find an appropriate graphing window and produce a graph of the polar curve.

(a) $r = \sin 6\theta$ **(b)** $r = 1 - 2\cos\theta$ **(c)** $r = 4\sin\theta$

SOLUTION

For all these graphs, set your calculator to POLAR mode.

(a) First we find the window. Notice that $|r| = |\sin 6\theta| \leq 1$ for all θ, so points on the graph are all within 1 unit from the pole. We want a window at least as large as $[-1, 1]$ by $[-1, 1]$, but we choose the window $[-1.5, 1.5]$ by $[-1, 1]$ in order to keep the *aspect ratio* close to the screen dimensions, which have a ratio of 3:2. We choose a θ-range of $0 \leq \theta \leq 2\pi$ to get a full rotation around the graph, after which we know that $\sin 6\theta$ will repeat the same graph periodically. Choose θ step $= 0.05$. The result is shown in Figure 10.22a.

(b) In this graph we notice that $|r| = |1-2\cos\theta| \leq 3$, so we choose $[-3, 3]$ for our y-range and, to get the right aspect ratio, $[-4.5, 4.5]$ for our x-range. Due to the cosine's period, $0 \leq \theta \leq 2\pi$ again suffices for our θ-range. The graph is shown in Figure 10.22b.

(c) Since $|r| = |4\sin\theta| \leq 4$, we choose $[-4, 4]$ for our y-range and $[-6, 6]$ for our x-range. Due to the sine's period, $0 \leq \theta \leq 2\pi$ again suffices for our θ-range. The graph is shown in Figure 10.22c.

Now try Exercise 13.

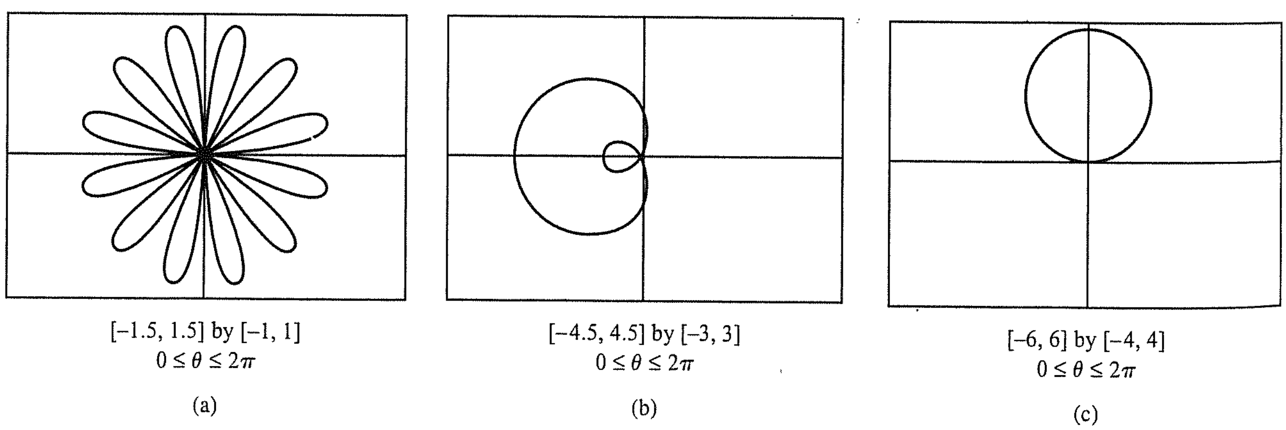


Figure 10.22 The graphs of the three polar curves in Example 3. The curves are (a) a 12-petaled rose, (b) a limaçon, and (c) a circle.

A Rose is a Rose

The graph in Figure 10.22a is called a 12-petaled rose, because it looks like a flower and some flowers are roses. The graph in Figure 10.22b is called a limaçon (LEE-ma-sohn) from an old French word for *snail*. We will have more names for you at the end of the section.

With a little experimentation, it is possible to improve on the "safe" windows we chose in Example 3 (at least in parts (b) and (c)), but it is always a good idea to keep a 3:2 ratio of the x-range to the y-range so that shapes do not become distorted. Also, an astute observer may have noticed that the graph in part (c) was traversed *twice* as θ went from 0 to 2π, so a range of $0 \leq \theta \leq \pi$ would have sufficed to produce the entire graph. From 0 to π, the circle is swept out by positive r values; then from π to 2π, the same circle is swept out by negative r values.

Although the graph in Figure 10.22c certainly looks like a circle, how can we tell for sure that it really is? One way is to convert the polar equation to a Cartesian equation and verify that it is the equation of a circle. Trigonometry gives us a simple way to convert polar equations to rectangular equations and vice versa.

Polar–Rectangular Conversion Formulas

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

EXAMPLE 4 Converting Polar to Rectangular

Use the polar–rectangular conversion formulas to show that the polar graph of $r = 4\sin\theta$ is a circle.

SOLUTION

To facilitate the substitutions, multiply both sides of the original equation by r. (This could introduce extraneous solutions with $r = 0$, but the pole is the only such point, and we notice that it is already on the graph.)

$$r = 4\sin\theta$$

$$r^2 = 4r\sin\theta \quad \text{Multiply by } r.$$

$$x^2 + y^2 = 4y \quad \text{Polar–rectangular conversion}$$

$$x^2 + y^2 - 4y = 0$$

$$x^2 + y^2 - 4y + 4 = 4 \quad \text{Completing the square}$$

$$x^2 + (y-2)^2 = 2^2 \quad \text{Circle in standard form}$$

Sure enough, the graph is a circle centered at (0, 2) with radius 2. ***Now try Exercise 25.***

The polar–rectangular conversion formulas also reveal the calculator's secret to polar graphing: It is really just parametric graphing with θ as the parameter.

Parametric Equations of Polar Curves

The polar graph of $r = f(\theta)$ is the curve defined parametrically by:

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

EXPLORATION 1 **Graphing Polar Curves Parametrically**

Switch your grapher to parametric mode and enter the equations

$$x = \sin(6t)\cos t$$

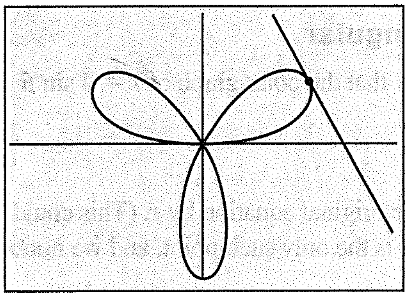
$$y = \sin(6t)\sin t.$$

1. Set an appropriate window and see if you can reproduce the polar graph in Figure 10.22a.
2. Then produce the graphs in Figures 10.22b and 10.22c in the same way.

Slopes of Polar Curves

Since polar curves are drawn in the xy-plane, the *slope* of a polar curve is still the slope of the tangent line, which is dy/dx. The polar–rectangular conversion formulas enable us to write x and y as functions of θ, so we can find dy/dx as we did with parametrically defined functions:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$



[−3, 3] by [−2, 2]
$0 \le \theta \le \pi$

Figure 10.23 The 3-petaled rose curve $r = 2\sin 3\theta$. Example 5 shows how to find the tangent line to the curve at $\theta = \pi/6$.

EXAMPLE 5 Finding Slope of a Polar Curve

Find the slope of the rose curve $r = 2\sin 3\theta$ at the point where $\theta = \pi/6$ and use it to find the equation of the tangent line (Figure 10.23).

SOLUTION

The slope is

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{\pi}{6}} = \left.\frac{\frac{d}{d\theta}(2\sin 3\theta \sin\theta)}{\frac{d}{d\theta}(2\sin 3\theta \cos\theta)}\right|_{\theta=\frac{\pi}{6}}.$$

This expression can be computed by hand, but it is an excellent candidate for your calculator's numerical derivative functionality (Section 3.2). NDERIV quickly gives an answer of -1.732050808, which you might recognize as $-\sqrt{3}$.

When $\theta = \pi/6$,

$$x = 2\sin(\pi/2)\cos(\pi/6) = \sqrt{3} \quad \text{and} \quad y = 2\sin(\pi/2)\sin(\pi/6) = 1.$$

So the tangent line has equation $y - 1 = -\sqrt{3}(x - \sqrt{3})$. ***Now try Exercise 39.***

Areas Enclosed by Polar Curves

We would like to be able to use numerical integration to find areas enclosed by polar curves just as we did with curves defined by their rectangular coordinates. Converting the equations to rectangular coordinates is not a reasonable option for most polar curves, so we would like to have a formula involving small changes in θ rather than small changes in x. While a small change Δx produces a thin *rectangular* strip of area, a small change $\Delta\theta$ produces a thin *circular sector* of area (Figure 10.24).

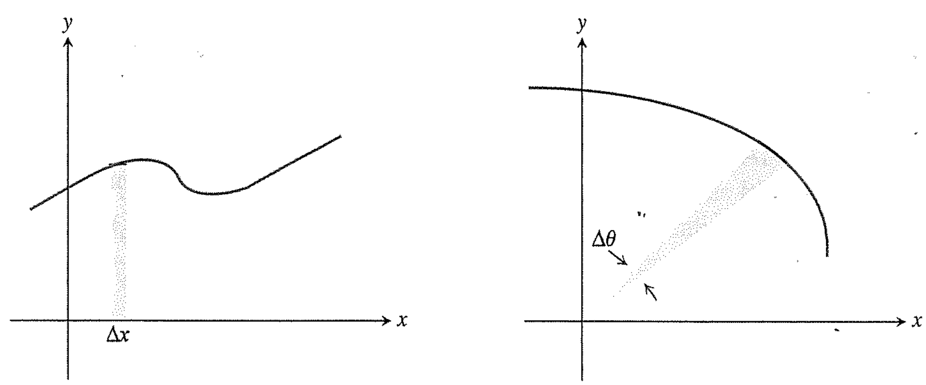


Figure 10.24 A small change in x produces a rectangular strip of area, while a small change in θ produces a thin *sector* of area.

Recall from geometry that the area of a sector of a circle is $\frac{1}{2}r^2\theta$, where r is the radius and θ is the central angle measured in radians. If we replace θ by the differential $d\theta$, we get the **area differential** $dA = \frac{1}{2}r^2 d\theta$ (Figure 10.25), which is exactly the quantity that we need to integrate to get an area in polar coordinates.

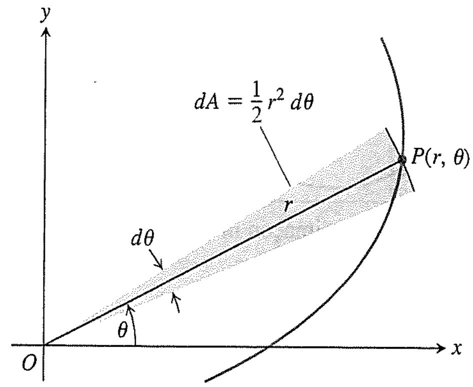


Figure 10.25 The area differential dA.

Area in Polar Coordinates

The area of the region between the origin and the curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2\, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)\right)^2 d\theta.$$

EXAMPLE 6 Finding Area

Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

SOLUTION

We graph the cardioid (Figure 10.26) and determine that the *radius OP* sweeps out the region exactly once as θ runs from 0 to 2π.

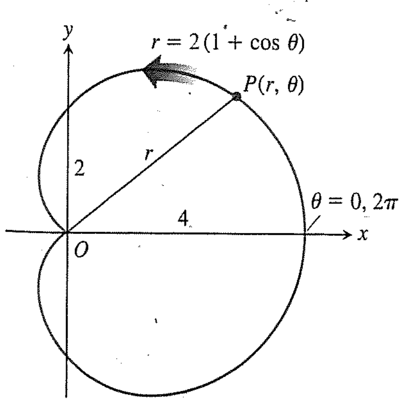


Figure 10.26 The cardioid in Example 6.

Solve Analytically The area is therefore

$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2\, d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2\, d\theta$$

$$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} \left(2 + 4\cos\theta + 2\frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta)\, d\theta$$

$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi.$$

Support Numerically NINT $(2(1 + \cos\theta)^2, \theta, 0, 2\pi) = 18.84955592$, which agrees with 6π to eight decimal places.

Now try Exercise 43.

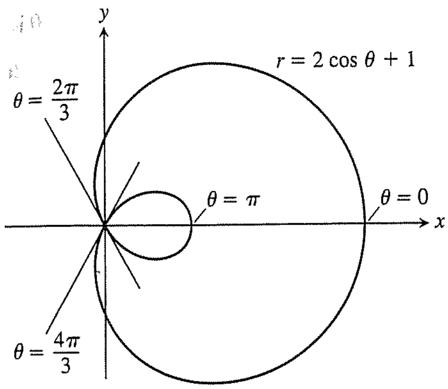


Figure 10.27 The limaçon in Example 7.

EXAMPLE 7 Finding Area

Find the area inside the smaller loop of the limaçon $r = 2\cos\theta + 1$.

SOLUTION

After watching the grapher generate the curve over the interval $0 \le \theta \le 2\pi$ (Figure 10.27), we see that the smaller loop is traced by the point (r, θ) as θ increases from $\theta = 2\pi/3$ to $\theta = 4\pi/3$ (the values for which $r = 2\cos\theta + 1 = 0$). The area we seek is

$$A = \int_{2\pi/3}^{4\pi/3} \frac{1}{2} r^2\, d\theta = \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (2\cos\theta + 1)^2\, d\theta.$$

Solve Numerically

$$\frac{1}{2}\text{NINT}\left((2\cos\theta + 1)^2, \theta, 2\pi/3, 4\pi/3\right) \approx 0.544.$$

Now try Exercise 47.

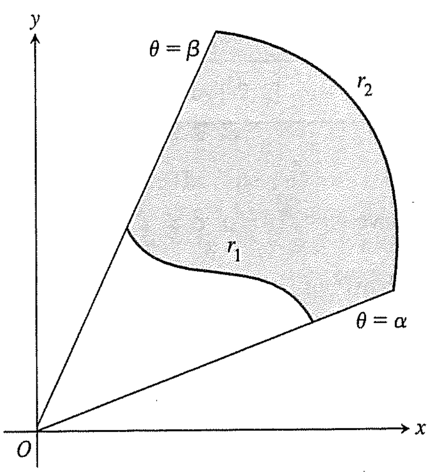


Figure 10.28 The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

To find the area of a region like the one in Figure 10.28, which lies between two polar curves $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we subtract the integral of $(1/2)r_1^2$ from the integral of $(1/2)r_2^2$. This leads to the following formula.

Area Between Polar Curves

The area of the region between $r_1(\theta)$ and $r_2(\theta)$ for $\alpha \le \theta \le \beta$ is

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2\, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2\, d\theta = \int_\alpha^\beta \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta.$$

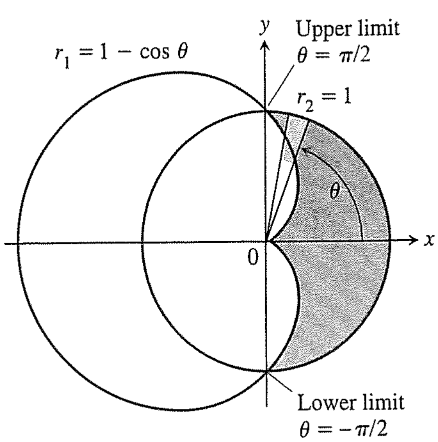


Figure 10.29 The region and limits of integration in Example 8.

EXAMPLE 8 Finding Area Between Curves

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

SOLUTION

The region is shown in Figure 10.29. The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and θ runs from $-\pi/2$ to $\pi/2$. Using the formula for the area between polar curves, the area is

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$

$$= 2\int_0^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad \text{Symmetry}$$

$$= \int_0^{\pi/2} \left(1 - (1 - 2\cos\theta + \cos^2\theta)\right) d\theta$$

$$= \int_0^{\pi/2} \left(2\cos\theta - \cos^2\theta\right) d\theta \approx 1.215. \qquad \text{Using NINT}$$

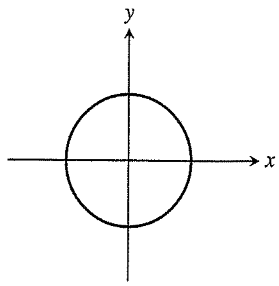
In case you are interested, the exact value is $2 - \pi/4$.

Now try Exercise 53.

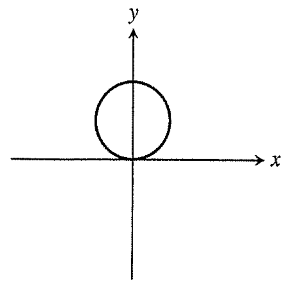
A SMALL POLAR GALLERY

Here are a few of the more common polar graphs and the θ-intervals that can be used to produce them.

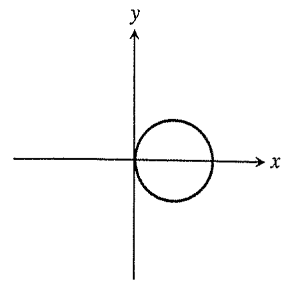
CIRCLES



$r = \text{constant}$
$0 \leq \theta \leq 2\pi$

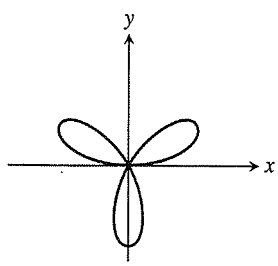


$r = a \sin \theta$
$0 \leq \theta \leq \pi$

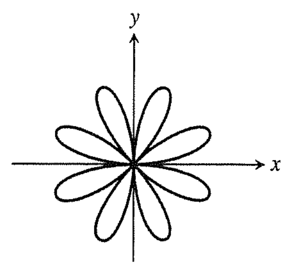


$r = a \cos \theta$
$0 \leq \theta \leq \pi$

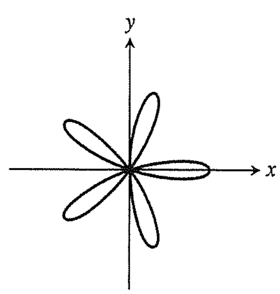
ROSE CURVES



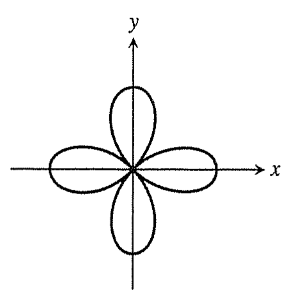
$r = a \sin n\theta$, n odd
$0 \leq \theta \leq \pi$
n petals
y-axis symmetry



$r = a \sin n\theta$, n even
$0 \leq \theta \leq 2\pi$
$2n$ petals
y-axis symmetry and
x-axis symmetry



$r = a \cos n\theta$, n odd
$0 \leq \theta \leq \pi$
n petals
x-axis symmetry

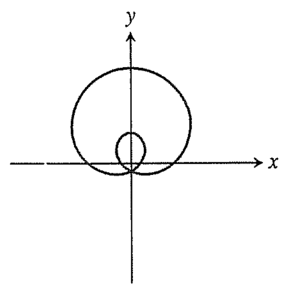


$r = a \cos n\theta$, n even
$0 \leq \theta \leq 2\pi$
$2n$ petals
y-axis symmetry and
x-axis symmetry

LIMAÇON CURVES

$r = a \pm b \sin \theta$ or $r = a \pm b \cos \theta$ with $a > 0$ and $b > 0$

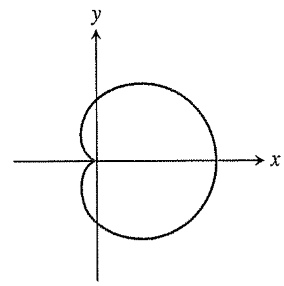
($r = a \pm b \sin \theta$ has y-axis symmetry; $r = a \pm b \cos \theta$ has x-axis symmetry.)



$\frac{a}{b} < 1$

$0 \leq \theta \leq 2\pi$

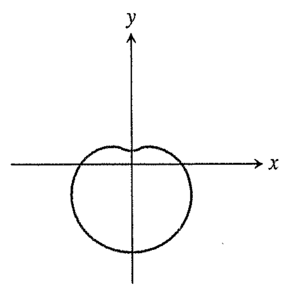
Limaçon with loop



$\frac{a}{b} = 1$

$0 \leq \theta \leq 2\pi$

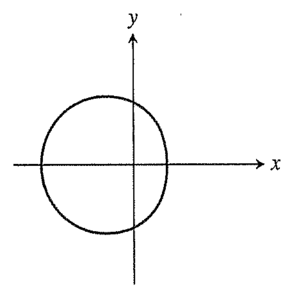
Cardioid



$1 < \frac{a}{b} < 2$

$0 \leq \theta \leq \pi$

Dimpled limaçon

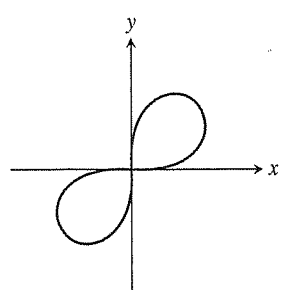


$\frac{a}{b} \geq 2$

$0 \leq \theta \leq 2\pi$

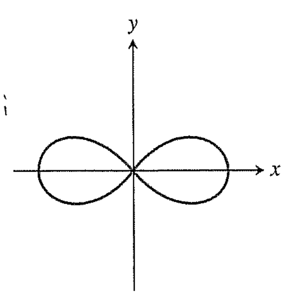
Convex limaçon

LEMNISCATE CURVES



$r^2 = a^2 \sin 2\theta$

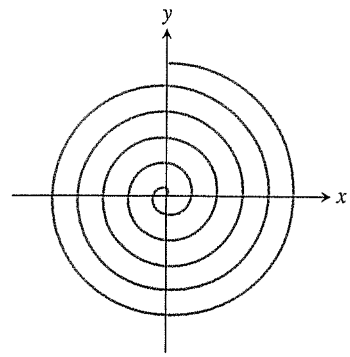
$0 \leq \theta \leq \pi$



$r^2 = a^2 \cos 2\theta$

$0 \leq \theta \leq \pi$

SPIRAL OF ARCHIMEDES



$r = \theta > 0$

Quick Review 10.3 *(For help, go to Sections 10.1 and 10.2.)*

1. Find the component form of a vector with magnitude 4 and direction angle 30°.

2. Find the area of a 30° sector of a circle of radius 6.

3. Find the area of a sector of a circle of radius 8 that has a central angle of $\pi/8$ radians.

4. Find the rectangular equation of a circle of radius 5 centered at the origin.

5. Explain how to use your calculator in function mode to graph the curve $x^2 + 3y^2 = 4$.

Exercises 6–10 refer to the parametrized curve

$$x = 3\cos t,\ y = 5\sin t,\ 0 \le t \le 2\pi.$$

6. Find dy/dx.

7. Find the slope of the curve at $t = 2$.

8. Find the points on the curve where the slope is zero.

9. Find the points on the curve where the slope is undefined.

10. Find the length of the curve from $t = 0$ to $t = \pi$.

Section 10.3 Exercises

In Exercises 1 and 2, plot each point with the given polar coordinates and find the corresponding rectangular coordinates.

1. **(a)** $(\sqrt{2}, \pi/4)$ **(b)** $(1, 0)$
(c) $(0, \pi/2)$ **(d)** $(-\sqrt{2}, \pi/4)$

2. **(a)** $(-3, 5\pi/6)$ **(b)** $(5, \tan^{-1}(4/3))$
(c) $(-1, 7\pi)$ **(d)** $(2\sqrt{3}, 2\pi/3)$

In Exercises 3 and 4, plot each point with the given rectangular coordinates and find two sets of corresponding polar coordinates.

3. **(a)** $(-1, 1)$ **(b)** $(1, -\sqrt{3})$
(c) $(0, 3)$ **(d)** $(-1, 0)$

4. **(a)** $(-\sqrt{3}, -1)$ **(b)** $(3, 4)$
(c) $(0, -2)$ **(d)** $(2, 0)$

In Exercises 5–10, graph the set of points whose polar coordinates satisfy the given equation.

5. $r = 3$ **6.** $r = -3$

7. $r^2 = 4$ **8.** $\theta = -\pi/4$

9. $|\theta| = \pi/6$ **10.** $r^2 + 8 = 6r$

In Exercises 11–20, find an appropriate window and use a graphing calculator to produce the polar curve. Then sketch the complete curve and identify the type of curve by name.

11. $r = 1 + \cos\theta$ **12.** $r = 2 - 2\cos\theta$

13. $r = 2\cos 3\theta$ **14.** $r = -3\sin 2\theta$

15. $r = 1 - 2\sin\theta$ **16.** $r = 3/2 + \cos\theta$

17. $r^2 = 4\cos 2\theta$ **18.** $r^2 = \sin 2\theta$

19. $r = 4\sin\theta$ **20.** $r = 3\cos\theta$

In Exercises 21–30, replace the polar equation by an equivalent Cartesian (rectangular) equation. Then identify or describe the graph.

21. $r = 4\csc\theta$ **22.** $r = -3\sec\theta$

23. $r\cos\theta + r\sin\theta = 1$ **24.** $r^2 = 1$

25. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$ **26.** $r^2 \sin 2\theta = 2$

27. $\cos^2\theta = \sin^2\theta$ **28.** $r^2 = -4r\cos\theta$

29. $r = 8\sin\theta$

30. $r = 2\cos\theta + 2\sin\theta$

In Exercises 31–38, find an appropriate window and use a graphing calculator to produce the polar curve. Then sketch the complete curve and identify the type of curve by name. (*Note*: You won't find these in the Polar Gallery.)

31. $r = \sec\theta \tan\theta$

32. $r = -\csc\theta \cot\theta$

33. $r = \dfrac{1}{1 + \cos\theta}$

34. $r = \dfrac{2}{1 - \sin\theta}$

35. $r = \dfrac{14}{5 + 9\cos\theta}$

36. $r = \dfrac{12}{8 + 6\cos\theta}$

37. $r = \dfrac{1}{1 - 0.8\cos\theta}$

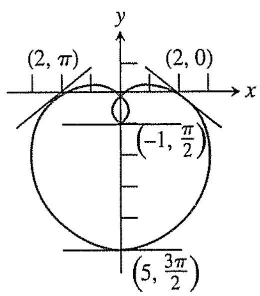
38. $r = \dfrac{1}{1 - 1.3\cos\theta}$

In Exercises 39–42, find the slope of the curve at each indicated point.

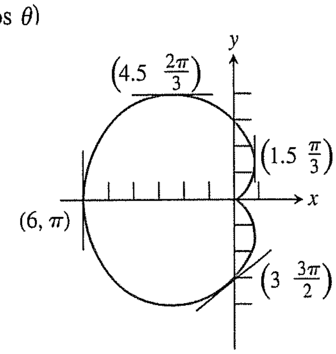
39. $r = -1 + \sin\theta$, $\theta = 0, \pi$

40. $r = \cos 2\theta$, $\theta = 0, \pm\pi/2, \pi$

41. $r = 2 - 3\sin\theta$



42. $r = 3(1 - \cos\theta)$



In Exercises 43–56, find the area of the region described.

43. inside the convex limaçon $r = 4 + 2\cos\theta$

44. inside the cardioid $r = 2 + 2\sin\theta$

45. inside one petal of the four-petaled rose $r = \cos 2\theta$

46. inside the eight-petaled rose $r = 2\sin 4\theta$

47. inside one loop of the lemniscate $r^2 = 4\cos 2\theta$

48. inside the six-petaled rose $r^2 = 2\sin 3\theta$

49. inside the dimpled limaçon $r = 3 - 2\cos\theta$

50. inside the inner loop of the limaçon $r = 2\sin\theta - 1$

51. shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$

52. shared by the circles $r = 1$ and $r = 2\sin\theta$

53. shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$

54. shared by the cardioids $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$

55. inside the circle $r = 2$ and outside the cardioid $r = 2(1 - \sin\theta)$

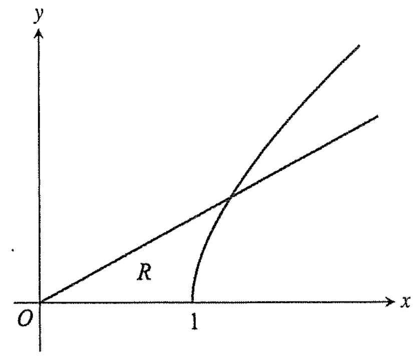
56. inside the four-petaled rose $r = 4\cos 2\theta$ and outside the circle $r = 2$

57. Sketch the polar curves $r = 3\cos\theta$ and $r = 1 + \cos\theta$ and find the area that lies inside the circle and outside the cardioid.

58. Sketch the polar curves $r = 2$ and $r = 2(1 - \sin\theta)$ and find the area that lies inside the circle and outside the cardioid.

59. Sketch the polar curve $r = 2\sin 3\theta$. Find the area enclosed by the curve and find the slope of the curve at the point where $\theta = \pi/4$.

60. The accompanying figure shows the parts of the graphs of the line $x = \frac{5}{3}y$ and the curve $x = \sqrt{1 + y^2}$ that lie in the first quadrant. Region R is enclosed by the line, the curve, and the x-axis.



(a) Set up and evaluate an integral expression with respect to y that gives the area of R.

(b) Show that the curve $x = \sqrt{1 + y^2}$ can be described in polar coordinates by $r^2 = \dfrac{1}{\cos^2\theta - \sin^2\theta}$.

(c) Use the polar equation in part (b) to set up an integral expression with respect to θ that gives the area of R.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

61. True or False There is exactly one point in the plane with polar coordinates (2, 2). Justify your answer.

62. True or False The total area enclosed by the 3-petaled rose $r = \sin 3\theta$ is $\int_0^{2\pi} \frac{1}{2}\sin^2 3\theta\, d\theta$. Justify your answer.

63. Multiple Choice The area of the region enclosed by the polar graph of $r = \sqrt{3 + \cos\theta}$ is given by which integral?

(A) $\int_0^{2\pi} \sqrt{3 + \cos\theta}\, d\theta$

(B) $\int_0^{\pi} \sqrt{3 + \cos\theta}\, d\theta$

(C) $2\int_0^{\pi/2} (3 + \cos\theta)\, d\theta$

(D) $\int_0^{\pi} (3 + \cos\theta)\, d\theta$

(E) $\int_0^{\pi/2} \sqrt{3 + \cos\theta}\, d\theta$

64. Multiple Choice The area enclosed by one petal of the 3-petaled rose $r = 4\cos(3\theta)$ is given by which integral?

(A) $16\int_{-\pi/3}^{\pi/3} \cos(3\theta)\, d\theta$

(B) $8\int_{-\pi/6}^{\pi/6} \cos(3\theta)\, d\theta$

(C) $8\int_{-\pi/3}^{\pi/3} \cos^2(3\theta)\, d\theta$

(D) $16\int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\, d\theta$

(E) $8\int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\, d\theta$

65. **Multiple Choice** If $a \neq 0$ and $\theta \neq 0$, all of the following must necessarily represent the same point in polar coordinates *except* which ordered pair?

(A) (a, θ) (B) $(-a, -\theta)$ (C) $(-a, \theta - \pi)$
(D) $(-a, \theta + \pi)$ (E) $(a, \theta - 2\pi)$

66. **Multiple Choice** Which of the following gives the slope of the polar curve $r = f(\theta)$ graphed in the xy-plane?

(A) $\frac{dr}{d\theta}$ (B) $\frac{dy}{d\theta}$ (C) $\frac{dx}{d\theta}$ (D) $\frac{dy/d\theta}{dx/d\theta}$ (E) $\frac{dy}{dx}\frac{dr}{d\theta}$

Explorations

67. ***Rotating Curves*** Let $r_1(\theta) = 3(1 - \cos\theta)$ and $r_2(\theta) = r_1(\theta - \alpha)$.

(a) Graph r_2 for $\alpha = \pi/6, \pi/4, \pi/3$, and $\pi/2$ and compare with the graph of r_1.

(b) Graph r_2 for $\alpha = -\pi/6, -\pi/4, -\pi/3$, and $-\pi/2$ and compare with the graph of r_1.

(c) Based on your observations in parts (a) and (b), describe the relationship between the graphs of $r_1 = f(\theta)$ and $r_2 = f(\theta - \alpha)$.

68. Let $r = \dfrac{2}{1 + k\cos\theta}$.

(a) Graph r in a square viewing window for $k = 0.1, 0.3, 0.5, 0.7$, and 0.9. Describe the graphs.

(b) Based on your observations in part (a), conjecture what happens to the graphs for $0 < k < 1$ and $k \to 0^+$.

69. Let $r = \dfrac{2}{1 + k\cos\theta}$.

(a) Graph r in a square viewing window for $k = 1.1, 1.3, 1.5, 1.7$, and 1.9. Describe the graphs.

(b) Based on your observations in part (a), conjecture what happens to the graphs for $k > 1$ and $k \to 1^+$.

70. Let $r = \dfrac{k}{1 + \cos\theta}$.

(a) Graph r in a square viewing window for $k = 1, 3, 5, 7$, and 9. Describe the graphs.

(b) Based on your observations in part (a), conjecture what happens to the graphs for $k > 0$ and $k \to 0^+$.

Extending the Ideas

71. ***Distance Formula*** Show that the distance between two points (r_1, θ_1) and (r_2, θ_2) in polar coordinates is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}.$$

72. ***Average Value*** If f is continuous, the average value of the polar coordinate r over the curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, with respect to θ is

$$r_{av} = \frac{1}{\beta - \alpha}\int_\alpha^\beta f(\theta)\,d\theta.$$

Use this formula to find the average value of r with respect to θ over the following curves ($a > 0$).

(a) the cardioid $r = a(1 - \cos\theta)$

(b) the circle $r = a$

(c) the circle $r = a\cos\theta$, $-\pi/2 \leq \theta \leq \pi/2$

73. ***Length of a Polar Curve*** The parametric form of the arc length formula (Section 10.1) gives the length of a polar curve as

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta.$$

Assuming that the necessary derivatives are continuous, show that the substitutions $x = r\cos\theta$ and $y = r\sin\theta$ transform this expression into

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

74. ***Length of a Cardioid*** Use the formula in Exercise 73 to find the length of the cardioid $r = 1 + \cos\theta$.

Quick Quiz for AP* Preparation: Sections 10.1–10.3

You may use a graphing calculator to solve the following problems.

1. **Multiple Choice** Which of the following is equal to the area of the region inside the polar curve $r = 2\cos\theta$ and outside the polar curve $r = \cos\theta$?

(A) $3\int_0^{\pi/2}\cos^2\theta\,d\theta$ (B) $3\int_0^{\pi}\cos^2\theta\,d\theta$

(C) $\frac{3}{2}\int_0^{\pi/2}\cos^2\theta\,d\theta$ (D) $3\int_0^{\pi/2}\cos\theta\,d\theta$

(E) $3\int_0^{\pi}\cos\theta\,d\theta$

2. **Multiple Choice** For what values of t does the curve given by the parametric equations $x = t^3 - t^2 - 1$ and $y = t^4 + 2t^2 - 8t$ have a vertical tangent?

(A) 0 only (B) 1 only
(C) 0 and 2/3 only (D) 0, 2/3, and 1
(E) No value

3. **Multiple Choice** The length of the path described by the parametric equations $x = t^2$ and $y = t$ from $t = 0$ to $t = 4$ is given by which integral?

(A) $\int_0^4\sqrt{4t + 1}\,dt$ (B) $2\int_0^4\sqrt{t^2 + 1}\,dt$ (C) $\int_0^4\sqrt{2t^2 + 1}\,dt$

(D) $\int_0^4\sqrt{4t^2 + 1}\,dt$ (E) $2\pi\int_0^4\sqrt{4t^2 + 1}\,dt$

4. **Free Response** A polar curve is defined by the equation $r = \theta + \sin 2\theta$ for $0 \leq \theta \leq \pi$.

(a) Find the area bounded by the curve and the x-axis.

(b) Find the angle θ that corresponds to the point on the curve where $x = -2$.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. How can this be seen from the graph?

(d) At what angle θ in the interval $0 \leq \theta \leq \pi/2$ is the curve farthest away from the origin? Justify your answer.