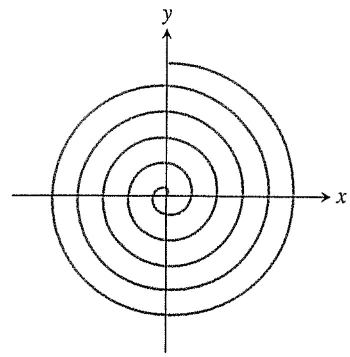
SPIRAL OF ARCHIMEDES



$r = \theta > 0$

Quick Review 10.3 *(For help, go to Sections 10.1 and 10.2.)*

1. Find the component form of a vector with magnitude 4 and direction angle 30°.
2. Find the area of a 30° sector of a circle of radius 6.
3. Find the area of a sector of a circle of radius 8 that has a central angle of $\pi/8$ radians.
4. Find the rectangular equation of a circle of radius 5 centered at the origin.
5. Explain how to use your calculator in function mode to graph the curve $x^2 + 3y^2 = 4$.

Exercises 6–10 refer to the parametrized curve

$$x = 3\cos t,\ y = 5\sin t,\ 0 \le t \le 2\pi.$$

6. Find dy/dx.
7. Find the slope of the curve at $t = 2$.
8. Find the points on the curve where the slope is zero.
9. Find the points on the curve where the slope is undefined.
10. Find the length of the curve from $t = 0$ to $t = \pi$.

Section 10.3 Exercises

In Exercises 1 and 2, plot each point with the given polar coordinates and find the corresponding rectangular coordinates.

1. **(a)** $(\sqrt{2}, \pi/4)$ **(b)** $(1, 0)$
   (c) $(0, \pi/2)$ **(d)** $(-\sqrt{2}, \pi/4)$
2. **(a)** $(-3, 5\pi/6)$ **(b)** $(5, \tan^{-1}(4/3))$
   (c) $(-1, 7\pi)$ **(d)** $(2\sqrt{3}, 2\pi/3)$

In Exercises 3 and 4, plot each point with the given rectangular coordinates and find two sets of corresponding polar coordinates.

3. **(a)** $(-1, 1)$ **(b)** $(1, -\sqrt{3})$
   (c) $(0, 3)$ **(d)** $(-1, 0)$
4. **(a)** $(-\sqrt{3}, -1)$ **(b)** $(3, 4)$
   (c) $(0, -2)$ **(d)** $(2, 0)$

In Exercises 5–10, graph the set of points whose polar coordinates satisfy the given equation.

5. $r = 3$
6. $r = -3$
7. $r^2 = 4$
8. $\theta = -\pi/4$
9. $|\theta| = \pi/6$
10. $r^2 + 8 = 6r$

In Exercises 11–20, find an appropriate window and use a graphing calculator to produce the polar curve. Then sketch the complete curve and identify the type of curve by name.

11. $r = 1 + \cos\theta$
12. $r = 2 - 2\cos\theta$
13. $r = 2\cos 3\theta$
14. $r = -3\sin 2\theta$
15. $r = 1 - 2\sin\theta$
16. $r = 3/2 + \cos\theta$
17. $r^2 = 4\cos 2\theta$
18. $r^2 = \sin 2\theta$
19. $r = 4\sin\theta$
20. $r = 3\cos\theta$

In Exercises 21–30, replace the polar equation by an equivalent Cartesian (rectangular) equation. Then identify or describe the graph.

21. $r = 4\csc\theta$
22. $r = -3\sec\theta$
23. $r\cos\theta + r\sin\theta = 1$
24. $r^2 = 1$
25. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$
26. $r^2\sin 2\theta = 2$
27. $\cos^2\theta = \sin^2\theta$
28. $r^2 = -4r\cos\theta$
29. $r = 8\sin\theta$
30. $r = 2\cos\theta + 2\sin\theta$

In Exercises 31–38, find an appropriate window and use a graphing calculator to produce the polar curve. Then sketch the complete curve and identify the type of curve by name. (*Note*: You won't find these in the Polar Gallery.)

31. $r = \sec\theta \tan\theta$

32. $r = -\csc\theta \cot\theta$

33. $r = \dfrac{1}{1 + \cos\theta}$

34. $r = \dfrac{2}{1 - \sin\theta}$

35. $r = \dfrac{14}{5 + 9\cos\theta}$

36. $r = \dfrac{12}{8 + 6\cos\theta}$

37. $r = \dfrac{1}{1 - 0.8\cos\theta}$

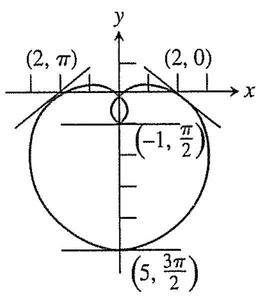
38. $r = \dfrac{1}{1 - 1.3\cos\theta}$

In Exercises 39–42, find the slope of the curve at each indicated point.

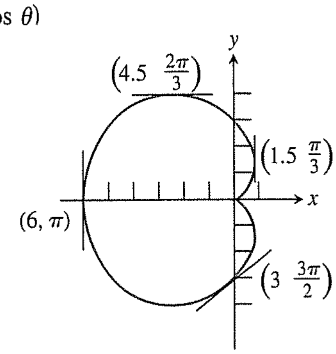
39. $r = -1 + \sin\theta,\ \theta = 0, \pi$

40. $r = \cos 2\theta,\ \theta = 0, \pm\pi/2, \pi$

41. $r = 2 - 3\sin\theta$



42. $r = 3(1 - \cos\theta)$



In Exercises 43–56, find the area of the region described.

43. inside the convex limaçon $r = 4 + 2\cos\theta$

44. inside the cardioid $r = 2 + 2\sin\theta$

45. inside one petal of the four-petaled rose $r = \cos 2\theta$

46. inside the eight-petaled rose $r = 2\sin 4\theta$

47. inside one loop of the lemniscate $r^2 = 4\cos 2\theta$

48. inside the six-petaled rose $r^2 = 2\sin 3\theta$

49. inside the dimpled limaçon $r = 3 - 2\cos\theta$

50. inside the inner loop of the limaçon $r = 2\sin\theta - 1$

51. shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$

52. shared by the circles $r = 1$ and $r = 2\sin\theta$

53. shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$

54. shared by the cardioids $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$

55. inside the circle $r = 2$ and outside the cardioid $r = 2(1 - \sin\theta)$

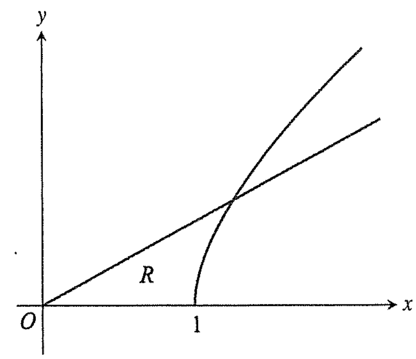
56. inside the four-petaled rose $r = 4\cos 2\theta$ and outside the circle $r = 2$

57. Sketch the polar curves $r = 3\cos\theta$ and $r = 1 + \cos\theta$ and find the area that lies inside the circle and outside the cardioid.

58. Sketch the polar curves $r = 2$ and $r = 2(1 - \sin\theta)$ and find the area that lies inside the circle and outside the cardioid.

59. Sketch the polar curve $r = 2\sin 3\theta$. Find the area enclosed by the curve and find the slope of the curve at the point where $\theta = \pi/4$.

60. The accompanying figure shows the parts of the graphs of the line $x = \frac{5}{3}y$ and the curve $x = \sqrt{1 + y^2}$ that lie in the first quadrant. Region R is enclosed by the line, the curve, and the x-axis.



(a) Set up and evaluate an integral expression with respect to y that gives the area of R.

(b) Show that the curve $x = \sqrt{1 + y^2}$ can be described in polar coordinates by $r^2 = \dfrac{1}{\cos^2\theta - \sin^2\theta}$.

(c) Use the polar equation in part (b) to set up an integral expression with respect to θ that gives the area of R.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

61. True or False There is exactly one point in the plane with polar coordinates (2, 2). Justify your answer.

62. True or False The total area enclosed by the 3-petaled rose $r = \sin 3\theta$ is $\int_0^{2\pi} \frac{1}{2}\sin^2 3\theta\, d\theta$. Justify your answer.

63. Multiple Choice The area of the region enclosed by the polar graph of $r = \sqrt{3 + \cos\theta}$ is given by which integral?

(A) $\int_0^{2\pi} \sqrt{3 + \cos\theta}\, d\theta$ (B) $\int_0^{\pi} \sqrt{3 + \cos\theta}\, d\theta$

(C) $2\int_0^{\pi/2} (3 + \cos\theta)\, d\theta$ (D) $\int_0^{\pi} (3 + \cos\theta)\, d\theta$

(E) $\int_0^{\pi/2} \sqrt{3 + \cos\theta}\, d\theta$

64. Multiple Choice The area enclosed by one petal of the 3-petaled rose $r = 4\cos(3\theta)$ is given by which integral?

(A) $16\int_{-\pi/3}^{\pi/3} \cos(3\theta)\, d\theta$ (B) $8\int_{-\pi/6}^{\pi/6} \cos(3\theta)\, d\theta$

(C) $8\int_{-\pi/3}^{\pi/3} \cos^2(3\theta)\, d\theta$ (D) $16\int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\, d\theta$

(E) $8\int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\, d\theta$

65. **Multiple Choice** If $a \neq 0$ and $\theta \neq 0$, all of the following must necessarily represent the same point in polar coordinates *except* which ordered pair?

(A) (a, θ) (B) $(-a, -\theta)$ (C) $(-a, \theta - \pi)$
(D) $(-a, \theta + \pi)$ (E) $(a, \theta - 2\pi)$

66. **Multiple Choice** Which of the following gives the slope of the polar curve $r = f(\theta)$ graphed in the xy-plane?

(A) $\frac{dr}{d\theta}$ (B) $\frac{dy}{d\theta}$ (C) $\frac{dx}{d\theta}$ (D) $\frac{dy/d\theta}{dx/d\theta}$ (E) $\frac{dy}{dx}\frac{dr}{d\theta}$

Explorations

67. ***Rotating Curves*** Let $r_1(\theta) = 3(1 - \cos\theta)$ and $r_2(\theta) = r_1(\theta - \alpha)$.

(a) Graph r_2 for $\alpha = \pi/6, \pi/4, \pi/3$, and $\pi/2$ and compare with the graph of r_1.

(b) Graph r_2 for $\alpha = -\pi/6, -\pi/4, -\pi/3$, and $-\pi/2$ and compare with the graph of r_1.

(c) Based on your observations in parts (a) and (b), describe the relationship between the graphs of $r_1 = f(\theta)$ and $r_2 = f(\theta - \alpha)$.

68. Let $r = \frac{2}{1 + k\cos\theta}$.

(a) Graph r in a square viewing window for $k = 0.1, 0.3, 0.5, 0.7$, and 0.9. Describe the graphs.

(b) Based on your observations in part (a), conjecture what happens to the graphs for $0 < k < 1$ and $k \to 0^+$.

69. Let $r = \frac{2}{1 + k\cos\theta}$.

(a) Graph r in a square viewing window for $k = 1.1, 1.3, 1.5, 1.7$, and 1.9. Describe the graphs.

(b) Based on your observations in part (a), conjecture what happens to the graphs for $k > 1$ and $k \to 1^+$.

70. Let $r = \frac{k}{1 + \cos\theta}$.

(a) Graph r in a square viewing window for $k = 1, 3, 5, 7$, and 9. Describe the graphs.

(b) Based on your observations in part (a), conjecture what happens to the graphs for $k > 0$ and $k \to 0^+$.

Extending the Ideas

71. ***Distance Formula*** Show that the distance between two points (r_1, θ_1) and (r_2, θ_2) in polar coordinates is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}.$$

72. ***Average Value*** If f is continuous, the average value of the polar coordinate r over the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, with respect to θ is

$$r_{av} = \frac{1}{\beta - \alpha}\int_\alpha^\beta f(\theta)\,d\theta.$$

Use this formula to find the average value of r with respect to θ over the following curves ($a > 0$).

(a) the cardioid $r = a(1 - \cos\theta)$

(b) the circle $r = a$

(c) the circle $r = a\cos\theta$, $-\pi/2 \le \theta \le \pi/2$

73. ***Length of a Polar Curve*** The parametric form of the arc length formula (Section 10.1) gives the length of a polar curve as

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta.$$

Assuming that the necessary derivatives are continuous, show that the substitutions $x = r\cos\theta$ and $y = r\sin\theta$ transform this expression into

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

74. ***Length of a Cardioid*** Use the formula in Exercise 73 to find the length of the cardioid $r = 1 + \cos\theta$.

Quick Quiz for AP* Preparation: Sections 10.1–10.3

You may use a graphing calculator to solve the following problems.

1. **Multiple Choice** Which of the following is equal to the area of the region inside the polar curve $r = 2\cos\theta$ and outside the polar curve $r = \cos\theta$?

(A) $3\int_0^{\pi/2} \cos^2\theta\,d\theta$ (B) $3\int_0^{\pi} \cos^2\theta\,d\theta$

(C) $\frac{3}{2}\int_0^{\pi/2} \cos^2\theta\,d\theta$ (D) $3\int_0^{\pi/2} \cos\theta\,d\theta$

(E) $3\int_0^{\pi} \cos\theta\,d\theta$

2. **Multiple Choice** For what values of t does the curve given by the parametric equations $x = t^3 - t^2 - 1$ and $y = t^4 + 2t^2 - 8t$ have a vertical tangent?

(A) 0 only (B) 1 only

(C) 0 and 2/3 only (D) 0, 2/3, and 1

(E) No value

3. **Multiple Choice** The length of the path described by the parametric equations $x = t^2$ and $y = t$ from $t = 0$ to $t = 4$ is given by which integral?

(A) $\int_0^4 \sqrt{4t + 1}\,dt$ (B) $2\int_0^4 \sqrt{t^2 + 1}\,dt$ (C) $\int_0^4 \sqrt{2t^2 + 1}\,dt$

(D) $\int_0^4 \sqrt{4t^2 + 1}\,dt$ (E) $2\pi\int_0^4 \sqrt{4t^2 + 1}\,dt$

4. **Free Response** A polar curve is defined by the equation $r = \theta + \sin 2\theta$ for $0 \le \theta \le \pi$.

(a) Find the area bounded by the curve and the x-axis.

(b) Find the angle θ that corresponds to the point on the curve where $x = -2$.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. How can this be seen from the graph?

(d) At what angle θ in the interval $0 \le \theta \le \pi/2$ is the curve farthest away from the origin? Justify your answer.