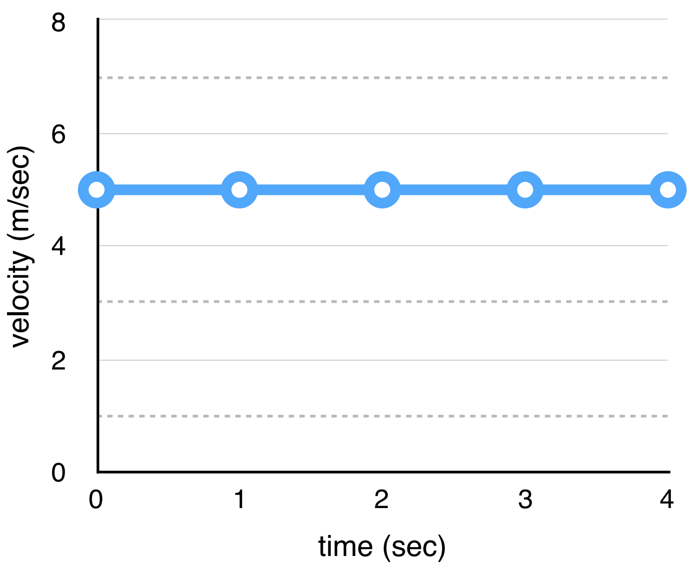
Name: Date: Per:

# Change & Rates of Change

Two concepts that physicist tend to trouble themselves over more than any others are how much a thing changes and at what rate that something changed. Much of physics boils down to answering those two questions. As such, we've come up with some clever tools to help us answer those questions. Those tools are called *integrals* and *derivatives*. Both those tools use mathematical expression to represent qualities that can be found on a graph.

You're probably familiar with the expression $d = r\,t$ (*distance equals rate times time*), but we can also use graphs to package together this (and more) information! Let's graph that "rate" on a velocity vs. time graph:



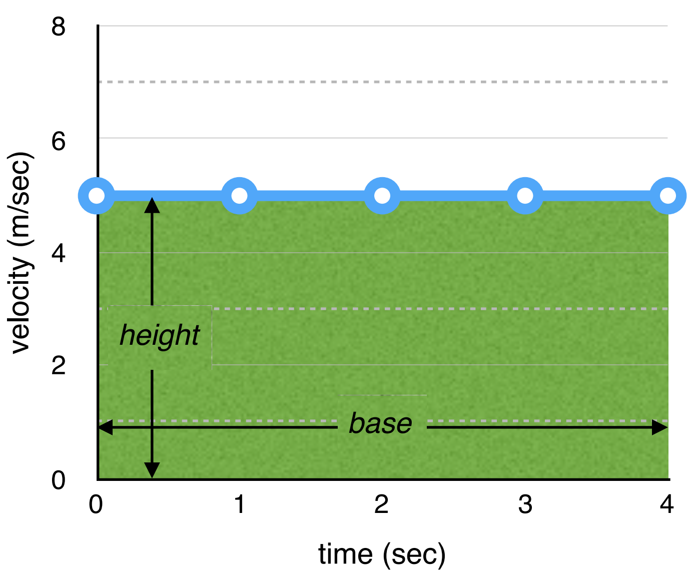

This might be a runner jogging at 5 meter per second for 4 seconds, it might be a bird flying at the same rate — it doesn't matter. Either way, our goal is to figure out how far they traveled.

We can already guess what the answer should be:

$d = r\,t$

$d$ = (5 m/sec)(4 sec)

$d$ = 20 meters




At first glance, it may not seem obvious where to find that distance represented on the graph, but in fact it's been there all along. It's the area under the graph!

In this case, we're looking at the area of a rectangle, which we know is *base times height*

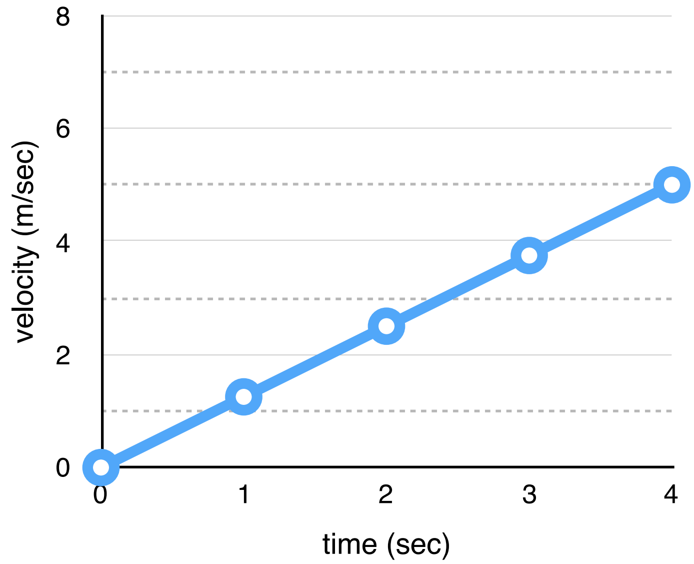
(*base*)×(*height*)

= (4 sec)(5 m/sec)

= 20 meters

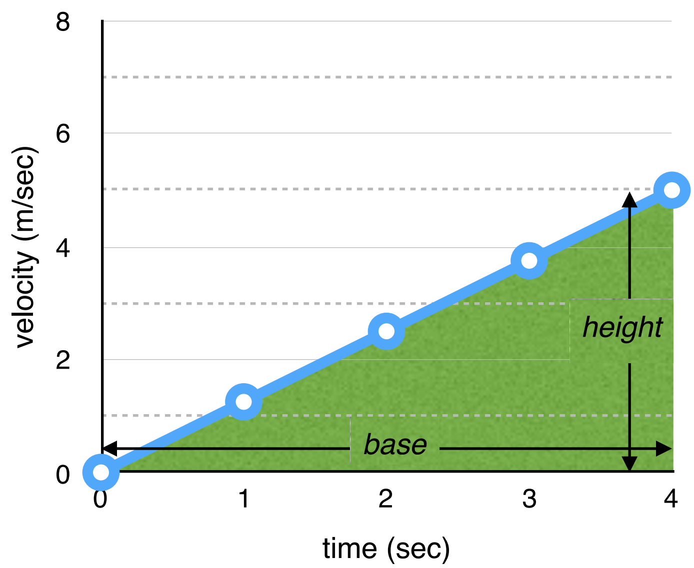
We get the same answer we already knew we should!

That trick of finding the area under the graph may seem trivial, but it becomes more useful when we look at an example where the velocity is *changing*.




In this case, our runner (or bird or car or whatever) starts from rest and gradually speeds up to 5 m/sec. In a case like this, $d = r\,t$ can't help us find the distance traveled because that only works for a *constant* speed.

However, using the area under the graph to find the distance traveled works exactly the same as before!




Now we're not trying to find the area of a rectangle but want the area of a triangle instead. But that's easy, too! The area of a triangle equals ½ (*base*) × (*height*)

½ (*base*)×(*height*)

= ½ (4 sec)(5 m/sec)

= 10 meters

So, using the area under the graph, we deduce that our jogger traveled 10 meters.

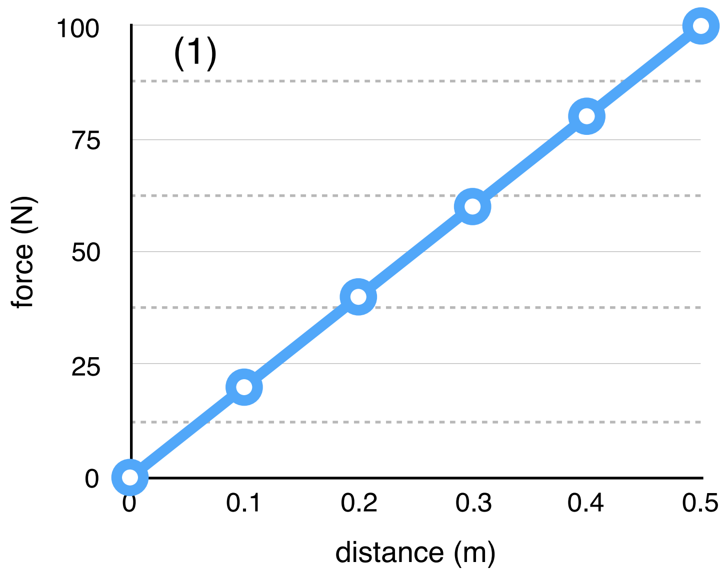
Physicists and mathematicians call this approach *integrating*, but that's just a fancy word for finding the area under the graph. Using that kind of terminology, we say that distance is equal to velocity integrated over time. As with anything in math, we have symbols we use to represent this idea:

$$d = \int v \, dt$$

The symbol "$\int$" is called an *integral*, and it tells you to find the area under the graph, which in this case is a velocity vs. time graph. A "*d*" gets attached to whatever variable is on the *x*-axis to remind us which variable belongs on which axis (it represents *change*, but we'll come back to that later).

# Practice

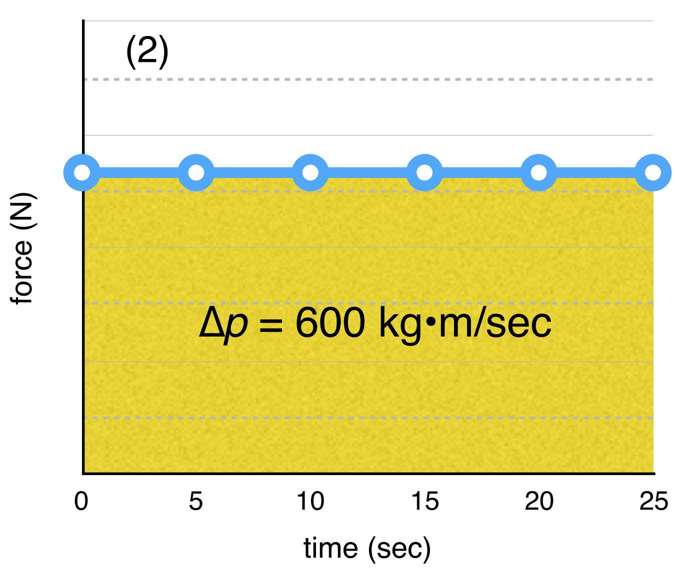
Use integration to deduce the desired quantity. Don't panic if you've never seen these equations before — the rules are all the same. Don't worry about units.




$$W = \int F\, dr$$

(*work* = *force* integrated over *distance*)

Find *work*:

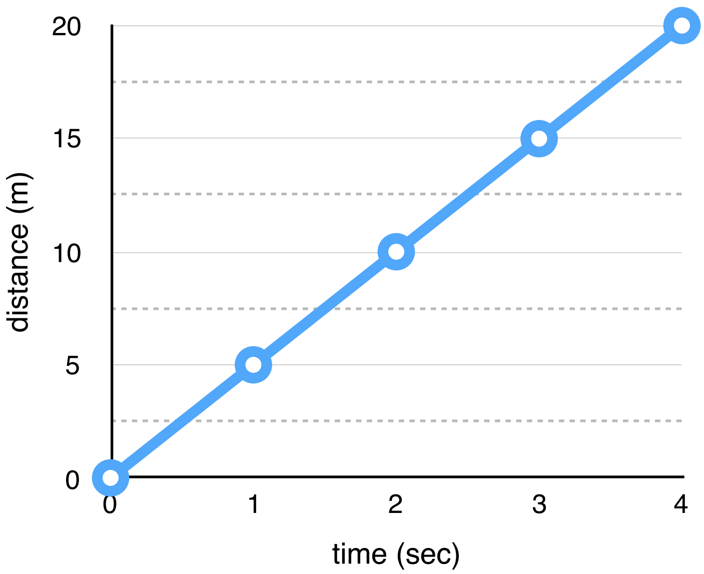



$$\Delta p = \int F\, dt$$

(*change in momentum* = *force* integrated over *time*)

Find the *force*:

Physicists also concern themselves with *rates of change*. As you change the independent variable, how much does the dependent variable change? Do small changes in the independent correspond to big or small changes in the dependent? Graphs can help us here, as well.

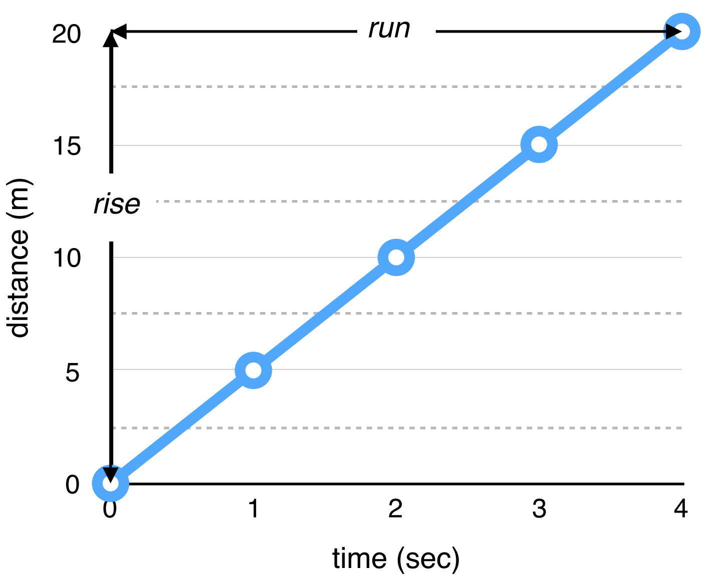



Let's return to $d = r\,t$, but this time let's plot out the distance vs. time graph.

We know velocity represents the rate at which distance changes, but can we deduce the velocity from the graph? Of course!

In a graph, rate of change is represented by the slope!

$$slope = \frac{rise}{run}$$




So by calculating the slope, we will have found the velocity.

$$\frac{rise}{run}$$

$$= \frac{(20\text{ m})}{(4\text{ sec})}$$

$$= 5\text{ m/sec}$$

So we deduce that our jogger is running at a speed of 5 m/sec.

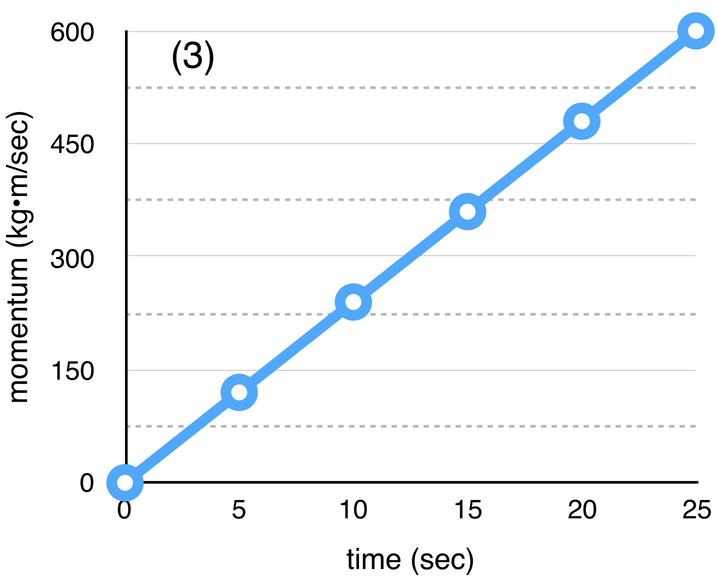
Physicists and mathematicians call the slope of a graph the *derivative*. Again, that's just fancy technical language, but using that kind of terminology, we'd say that velocity is equal to the time derivative of position (remember, distance just tells you how much your position changed, so the two are related). Once again, we introduce new math symbology to represent this idea:

$$v = \frac{dx}{dt}$$

Here I chose to use "*x*" to represent position. The *d*'s represent change, so the change in position divided by the change in time is the rate at which position changes with time. That's velocity!

## Practice

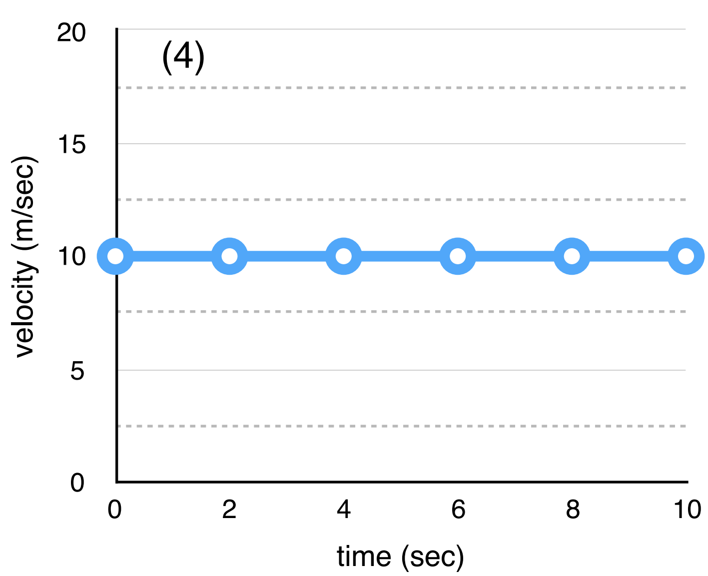
Use the derivative to deduce the desired quantity. Like before, don't panic if you've never seen these equations before — the rules are all the same. Don't worry about units.




$$F = \frac{dp}{dt}$$

(*force* = the rate at which *momentum* changes with *time*)

Find *force*:




$$a = \frac{dv}{dt}$$

(*acceleration* = the rate at which *velocity* changes with *time*)

Find *acceleration*: